Differential Geometry

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This document is the study note of the book [1].

1 Differentiable Manifolds

1.1 Tangent Spaces

Suppose M is an m-dimensional smooth manifold. Fix a point $p \in M$. Denote the set of all C^∞ functions defined in a neighborhood of p by C_p^∞. Define a relation $\sim$ in C_p^∞ as follows. Suppose $f, g \in C_p^\infty$. Then $f \sim g$ if and only if there exists an open neighborhood H of the point p such that $f|_H = g|_H$. Obviously $\sim$ is an equivalence relation in C_p^∞. The equivalence class of f is denoted by $[f]$, called a C^∞**-germ** at p on M. Let

$$\mathcal{F}_p = C_p^\infty / \sim = \{[f] \mid f \in C_p^\infty\}.$$

Then $\mathcal{F}_p$ is a linear space over $\mathbb{R}$ with regular addition and scalar multiplication.

For a parametrized curve γ in M through a point p, there exists a positive number δ such that $\gamma : (-\delta, \delta) \to M$ is C^∞ with $\gamma(0) = p$. Denote the set of all these parametrized curves by Γ_p.

We introduce a pairing between Γ_p and $\mathcal{F}_p$ by letting

$$\langle \gamma, [f] \rangle = \left. \frac{\mathrm{d}(f \circ \gamma)}{\mathrm{d}t} \right|_{t=0}$$

for each $\gamma \in \Gamma_p$ and $[f] \in \mathcal{F}_p$. This pairing is well-defined and linear in the second variable. Let

$$\mathcal{H}_p = \{[f] \in \mathcal{F}_p \mid \langle \gamma, [f] \rangle = 0, \forall \gamma \in \Gamma_p\}$$

be a linear subspace of $\mathcal{F}_p$.

Theorem 1.1.1 Suppose $[f] \in \mathcal{F}_p$. For a chart (U, φ), let $F = f \circ \varphi^{-1}$ be a function from an open subset of $\mathbb{R}^m$ to $\mathbb{R}$. Then $[f] \in \mathcal{H}_p$ if and only if

$$\left.\frac{\partial F}{\partial x^i}\right|_{\varphi(p)} = 0, \quad 1 \le i \le m.$$

Definition 1.1.1 The quotient space $\mathcal{F}_p/\mathcal{H}_p$ is called the **cotangent space** of M at p, denoted by T_p^* or $T_p^*(M)$. The $\mathcal{H}_p$-equivalence class of the C^∞-germ $[f]$ is denoted by $(\mathrm{d}f)_p$, called a **cotangent vector** on M at p.

The cotangent space T_p^* is a linear space with the linear structure induced from $\mathcal{F}_p$.

Theorem 1.1.2 Suppose $f^1, f^2, \cdots, f^s \in C_p^\infty$ and $F(y^1, y^2, \cdots, y^s)$ is a smooth function in a neighborhood of $(f^1(p), f^2(p), \cdots, f^s(p)) \in \mathbb{R}^s$. Then $f = F(f^1, f^2, \cdots, f^s) \in C_p^\infty$ and

$$(\mathrm{d}f)_p = \sum_{k=1}^{s} \left[\frac{\partial F}{\partial y^k}(f^1(p), f^2(p), \cdots, f^s(p)) \cdot (\mathrm{d}f^k)_p \right].$$

Corollary 1.1.3 For any $f, g \in C_p^\infty, a \in \mathbb{R}$, we have

1. $(\mathrm{d}(f+g))_p = (\mathrm{d}f)_p + (\mathrm{d}g)_p$,
2. $(\mathrm{d}(af))_p = a \cdot (\mathrm{d}f)_p$, and
3. $(\mathrm{d}(fg))_p = f(p) \cdot (\mathrm{d}g)_p + g(p) \cdot (\mathrm{d}f)_p$.

Choose a chart (U, φ) and define local coordinates u^i by $u^i(p) = (\varphi(p))^i = x^i \circ \varphi(p), p \in U$, where x^i is the standard coordinate system of $\mathbb{R}^m$. Then $u^i \in C_p^\infty$ and $(\mathrm{d}u^i)_p \in T_p^*$, $1 \le i \le m$. Choose $\lambda_k \in \Gamma_p, 1 \le k \le m$ such that

$$u^i \circ \lambda_k(t) = u^i(p) + \delta_k^i t.$$

Then we have

$$\langle \lambda_k, [u^i] \rangle = \left.\frac{\mathrm{d}}{\mathrm{d}t}(u^i \circ \lambda_k(t))\right|_{t=0} = \delta_k^i.$$

Theorem 1.1.4 $\{(\mathrm{d}u^i)_p, 1 \le i \le m\}$ is a basis of T_p^*, called the **natural basis** of T_p^* with respect to the local coordinate system u^i. It then follows that $\dim T_p^* = m$.

Proof. By Theorem 1.1.2, for each $f \in C_p^\infty$, $(\mathrm{d}f)_p$ is a linear combination of the $(\mathrm{d}u^i)_p, 1 \le i \le m$.

If there exist real numbers $a_i, 1 \le i \le m$ such that

$$\sum_{i=1}^m a_i (\mathrm{d}u^i)_p = 0,$$

then for any $\gamma \in \Gamma_p$, we have

$$\left\langle \gamma, \sum_{i=1}^m a_i [u^i] \right\rangle = \sum_{i=1}^m a_i \left. \frac{\mathrm{d}(u^i \circ \gamma(t))}{\mathrm{d}t} \right|_{t=0} = 0.$$

Let $\gamma = \lambda_k$ and we will obtain $a_k = 0, 1 \le k \le m$, i.e. $\{(\mathrm{d}u^i)_p, 1 \le i \le m\}$ is linearly independent. Therefore it forms a basis for T_p^*. □

We can simply define the pairing between Γ_p and T_p^* by

$$\langle \gamma, (\mathrm{d}f)_p \rangle = \langle \gamma, [f] \rangle$$

for each $\gamma \in \Gamma_p$ and $(\mathrm{d}f)_p \in T_p^*$ after the definition of $\mathcal{H}_p$ and T_p^*. Define a relation $\sim$ on Γ_p as follows. Suppose $\gamma, \gamma' \in \Gamma_p$. Then $\gamma \sim \gamma'$ if and only if for any $(\mathrm{d}f)_p \in T_p^*$,

$$\langle \gamma, (\mathrm{d}f)_p \rangle = \langle \gamma', (\mathrm{d}f)_p \rangle.$$

This is again an equivalence relation. Denote the equivalence class of γ by $[\gamma]$. We can then define

$$\langle [\gamma], (\mathrm{d}f)_p \rangle = \langle \gamma, (\mathrm{d}f)_p \rangle$$

without chance of confusion.

Theorem 1.1.5 The $\langle [\gamma], \cdot \rangle, \gamma \in \Gamma_p$ represent the totality of linear functionals on T_p^* and form its dual space, T_p, called the **tangent space** of M at p. Elements of the tangent space are called **tangent vectors** of M at p.

Proof. Suppose α is a linear functional on T_p^*. Let $\xi^i = \alpha(\mathrm{d}u^i)_p, 1 \le i \le m$. Choose $\gamma \in \Gamma_p$ such that

$$u^i(t) = u^i(p) + \xi^i t.$$

Then

$$\langle [\gamma], (\mathrm{d}f)_p \rangle = \sum_{i=1}^m \xi^i \left. \frac{\partial (f \circ \varphi^{-1})}{\partial u^i} \right|_{\varphi(p)} = \alpha(\mathrm{d}f)_p.$$

Therefore each linear functional on T_p^* can be expressed as $\langle[\gamma],\cdot\rangle$ for some $\gamma\in\Gamma_p$. Moreover, if $\langle[\gamma],\cdot\rangle$ and $\langle[\gamma'],\cdot\rangle$ are the same linear functionals on T_p^*, then $[\gamma]=[\gamma']$. Therefore, we can identify the space of $[\gamma],\gamma\in\Gamma_p$ with the dual space of T_p^*. □

The pairing $\langle X,(\mathrm{d}f)_p\rangle, X=[\gamma]\in T_p,(\mathrm{d}f)_p\in T_p^*$ is a bilinear map from $T_p\times T_p^*$ to $\mathbb{R}$. Noting that

$$\langle[\lambda_k],(\mathrm{d}u^i)_p\rangle=\delta_k^i,\quad 1\le i,k\le m,$$

$\{[\lambda_k],1\le k\le m\}$ is the basis of T_p dual to the basis $\{(\mathrm{d}u^i),1\le i\le m\}$ of T_p^*. The tangent vectors can also be seen as functions from C_p^∞ to $\mathbb{R}$. For a general $f\in C_p^\infty$, we have

$$\langle[\lambda_k],(\mathrm{d}f)_p\rangle=\left\langle[\lambda_k],\sum_{i=1}^m\left[\left(\frac{\partial f}{\partial u^i}\right)_p\cdot(\mathrm{d}u^i)_p\right]\right\rangle=\left(\frac{\partial f}{\partial u^k}\right)_p,$$

where $(\partial f/\partial u^i)_p$ means $(\partial(f\circ\varphi^{-1})/\partial x^i)_{\varphi(p)}$. Thus the $[\lambda_k]$ can be identified with the partial differential operators $(\partial/\partial u^k)_p$ on the space C_p^∞. The basis $\{(\partial/\partial u^k)_p,1\le k\le m\}$ is called the **natural basis** of T_p with respect to the local coordinate system u^i.

The lower index p of tangent and cotangent vectors can be suppressed for simplicity if there is no chance of confusion.

Definition 1.1.2 Suppose $X\in T_p,f\in C_p^\infty$. Then $(\mathrm{d}f)_p\in T_p^*$ is called the **differential** of f at the point p. Denote $Xf=\langle X,\mathrm{d}f\rangle$, then Xf is called the **directional derivative** of the function f along the vector X.

Theorem 1.1.6 Suppose $X\in T_p,f,g\in C_p^\infty,\alpha,\beta\in\mathbb{R}$. Then

1. $X(\alpha f+\beta g)=\alpha Xf+\beta Xg$;
2. $X(fg)=f(p)X(g)+g(p)X(g)$.

The above properties of tangent vectors also give an alternative definition of tangent vectors.

Smooth maps between smooth manifolds induce linear maps between tangent spaces and between cotangent spaces. Suppose $F:M\to N$ is a smooth map, $p\in M,q=F(p)\in N$. Define the map $F^*:T_q^*(N)\to T_p^*(M)$ by $F^*(\mathrm{d}f)=\mathrm{d}(f\circ F),\mathrm{d}f\in T_q^*(N)$. This is a well-defined linear map, called the **differential** of the map F. The adjoint of F^*, namely the map $F_*:T_p(M)\to T_q(N)$ given by

$$\langle F_*X,a\rangle=\langle X,F^*a\rangle,\quad X\in T_p(M),a\in T_q^*(N),$$

is called the **tangent map** induced by F.

Suppose u^i and v^α are local coordinates near p and q, respectively. Then the map F can be expressed near p by the functions

$$F^\alpha(u^1, \cdots, u^m) = v^\alpha \circ F(u^1, \cdots, u^m), \quad 1 \le \alpha \le n.$$

Then the action of F^* on the natural basis $\{(\mathrm{d}v^\alpha), 1 \le \alpha \le n\}$ is given by

$$F^*(\mathrm{d}v^\alpha) = \mathrm{d}F^\alpha = \sum_{i=1}^{m} \left(\frac{\partial F^\alpha}{\partial u^i}\right)_p \cdot \mathrm{d}u^i.$$

Hence the matrix representation of F^* in the natural bases $\{\mathrm{d}v^\alpha\}$ and $\{\mathrm{d}u^i\}$ is exactly the Jacobian matrix $\big((\partial F^\alpha/\partial u^i)_p\big)$. Similarly, the action of F_* on the natural basis $\{\partial/\partial u^i, 1 \le i \le m\}$ is given by

$$\begin{aligned}
\left\langle F_*\left(\frac{\partial}{\partial u^i}\right), \mathrm{d}v^\alpha \right\rangle &= \left\langle \frac{\partial}{\partial u^i}, F^*(\mathrm{d}v^\alpha) \right\rangle \\
&= \sum_{j=1}^{m} \left(\frac{\partial F^\alpha}{\partial u^j}\right)_p \left\langle \frac{\partial}{\partial u^i}, \mathrm{d}u^j \right\rangle \\
&= \left(\frac{\partial F^\alpha}{\partial u^i}\right)_p \\
&= \sum_{\beta=1}^{n} \left(\frac{\partial F^\beta}{\partial u^i}\right)_p \left\langle \frac{\partial}{\partial v^\beta}, \mathrm{d}v^\alpha \right\rangle \\
&= \left\langle \sum_{\beta=1}^{n} \left(\frac{\partial F^\beta}{\partial u^i}\right)_p \left(\frac{\partial}{\partial v^\beta}\right), \mathrm{d}v^\alpha \right\rangle,
\end{aligned}$$

i.e.

$$F_*\left(\frac{\partial}{\partial u^i}\right) = \sum_{\beta=1}^{n} \left(\frac{\partial F^\beta}{\partial u^i}\right)_p \left(\frac{\partial}{\partial v^\beta}\right).$$

Therefore the matrix representation of F_* in the natural bases $\{\partial/\partial u^i\}$ and $\{\partial/\partial v^\alpha\}$ is still the Jacobian matrix $\big((\partial F^\alpha/\partial u^i)_p\big)$.

1.2 Submanifolds

Using the Inverse Function Theorem for $\mathbb{R}^n$ and the local coordinate systems of manifolds, we can obtain the following generalization for manifolds.

Theorem 1.2.1 Suppose M and N are both n-dimensional smooth manifolds, and $f : M \to N$ is a smooth map. If at a point $p \in M$, the tangent map $f_* : T_p(M) \to T_{f(p)}(N)$ is an isomorphism, then there exists a neighborhood U of p in M such that $V = f(U)$ is a neighborhood of $f(p)$ in N and $f|_U : U \to V$ is a diffeomorphism.

If M is an m-dimensional manifold and N an n-dimensional manifold, $f : M \to N$ is smooth, and the tangent map f_* is injective at a point p, then f_* is said to be **nondegenerate** at p. In this case, we have $m \leq n$, and the rank of the Jacobian matrix of f at p is m.

Theorem 1.2.2 Suppose M is an m-dimensional manifold and N an n-dimensional manifold, $m < n$. If $f : M \to N$ is a smooth map and the tangent map f_* is nondegenerate at a point p in M, then there exist local coordinate systems $(U; u^i)$ near p and $(V; v^\alpha)$ near $q = f(p)$ such that $f(U) = V$, and the map $f|_U$ can be expressed by local coordinates as

$$\begin{cases} v^i(f(x)) = u^i(x), & 1 \leq i \leq m; \\ v^\gamma(f(x)) = 0, & m+1 \leq \gamma \leq n. \end{cases}$$

for each $x \in U$.

Proof. Take local coordinate systems $(U; u^i)$ and $(V; v^\alpha)$ at p and q, respectively, such that $u^i(p) = 0$ and $v^\alpha(q) = 0$. Since f_* is nondegenerate at p, we may assume that

$$\left.\frac{\partial(f^1, f^2, \cdots, f^m)}{\partial(u^1, u^2, \cdots, u^m)}\right|_{u^i=0} \neq 0.$$

Let $I_{n-m} = \{(w^{m+1}, \cdots, w^n) \mid |w^\gamma| \leq \delta, m+1 \leq \gamma \leq n\}$, where δ is a sufficiently small positive number. By suitably shrinking the neighborhood U, we can define a smooth map $\tilde{f} : U \times I_{n-m} \to V$ such that

$$\begin{cases} \tilde{f}^i(u^1, \cdots, u^m, w^{m+1}, \cdots, w^n) = f^i(u^1, \cdots, u^m), & 1 \leq i \leq m; \\ \tilde{f}^\gamma(u^1, \cdots, u^m, w^{m+1}, \cdots, w^n) = w^\gamma + f^\gamma(u^1, \cdots, u^m), & m+1 \leq \gamma \leq n. \end{cases}$$

The Jacobian matrix of $\tilde{f}$ at $(u^i, w^\gamma) = (0, 0)$ is nondegenerate. It follows by Theorem 1.2.1 that $\tilde{f}$ is a diffeomorphism ina neighborhood of $(0, 0)$. We may assume that $\tilde{f} : U \times I_{n-m} \to V$ is a diffeomorphism. Then there exists a coordinate system $\bar{v}^\alpha$ in the neighborhood V of q such that $\tilde{f}$ is expressed as

$$\begin{cases} \bar{v}^i(\tilde{f}(u^1, \cdots, u^m, w^{m+1}, \cdots, w^n)) = u^i, & 1 \leq i \leq m; \\ \bar{v}^\gamma(\tilde{f}(u^1, \cdots, u^m, w^{m+1}, \cdots, w^n)) = w^\gamma, & m+1 \leq \gamma \leq n. \end{cases}$$

Thus the local coordinate systems $(U; u^i)$ and $(V; \bar{v}^\alpha)$ are the desired. $\square$

Definition 1.2.1 Suppose M and N are smooth manifolds. If there is a smooth map $\varphi : M \to N$ such that the tangent map $\varphi_* : T_p(M) \to T_{\varphi(p)}(N)$ is nondegenerate at any point $p \in M$, then φ is called an **immersion**, and (φ, M) an **immersed submanifold** of N. Furthermore, if φ is also injective, then (φ, M) is called a **smooth submanifold**, or **imbedded submanifold**, of N.

By Theorem 1.2.2, an immersion is locally injective, but not necessarily so globally.

Example 1.2.1 Suppose U is an open subset of N. By restricting the smooth structure of N to U, we obtain a smooth structure on U, which makes U a smooth manifold with the same dimension as N. Let $\varphi : U \to N$ be the inclusion map, then (φ, U) becomes an imbedded submanifold of N, called an **open submanifold** of N.

Example 1.2.2 Suppose (φ, M) is a smooth submanifold of N. If

1. $\varphi(M)$ is a closed subset of N;
2. for any point $q \in \varphi(M)$, there exists a local coordinate system $(U; u^i)$ such that $\varphi(M) \cap U$ is defined by
$$u^{m+1} = u^{m+2} = \cdots = u^n = 0,$$
where $m = \dim M$,

then we call (φ, M) a **closed submanifold** of N.

For an imbedded submanifold (φ, M), since φ is injective, the differentiable structure of M can be transported to $\varphi(M)$, making $\varphi : M \to \varphi(M)$ a diffeomorphism. On the other hand, being a subset of N, $\varphi(M)$ has an induced topology from N. The topology on $\varphi(M)$ obtained from M through φ is not necessarily the same as the one induced from N.

Definition 1.2.2 Suppose (φ, M) is a smooth submanifold of N. If $\varphi : M \to \varphi(M) \subset N$ is a homeomorphism, then (φ, M) is called a **regular submanifold** of N, and φ is called a **regular imbedding** of M into N.

Theorem 1.2.3 Suppose (φ, M) is an m-dimensional submanifold of an n-dimensional smooth manifold of N. Then (φ, M) is a regular submanifold of N if and only if it is a closed submanifold of an open submanifold of N.

Proof. First we show that a closed submanifold (φ, M) of N is a regular submanifold. Choose an arbitrary point $p \in M$. There exists a local coordinate system $(V; v^\alpha)$ at the point $q = \varphi(p)$ in N such that $\varphi(M) \cap V$ is defined by

$$v^{m+1} = v^{m+2} = \cdots = v^n = 0.$$

Since φ is continuous, there exists a local coordinate system $(U; u^i)$ such that $\varphi(U) \subset V$. We may assume that $u^i(p) = 0, v^\alpha(q) = 0$, and $V = \{(v^1, \cdots, v^n) \mid |v^\alpha| < \delta\}$, where δ is a positive number. Thus $\varphi(U) \subset \varphi(M) \cap V$.

The goal is to prove that $\varphi^{-1} : \varphi(M) \subset N \to M$ is also continuous. The map $\varphi|_U$ can be expressed locally by

$$\begin{cases} v^i = \varphi(u^1, \cdots, u^m), \quad 1 \le i \le m; \\ v^\gamma = 0, \quad m+1 \le \gamma \le n. \end{cases}$$

Since φ_* is nondegenerate at p, the Jacobian

$$\left.\frac{\partial(\varphi^1, \varphi^2, \cdots, \varphi^m)}{\partial(u^1, u^2, \cdots, u^m)}\right|_{u^i=0} \neq 0.$$

By the Inverse Function Theorem, there exists δ_1 with $0 < \delta_1 < \delta$ such that there is an inverse function set

$$u^i = \psi^i(v^1, \cdots, v^m), \quad |v^i| < \delta_1$$

of the function set $(\varphi^1, \cdots, \varphi^m)$. Let $V_1 = \{(v^1, \cdots, v^n) \mid |v^\alpha| < \delta_1\}$, then the entire preimage of $\varphi(M) \cap V$ under φ is contained in U. Hence $\varphi : M \to \varphi(M) \subset N$ is a homeomorphism, which implies that (φ, M) is a regular submanifold of N.

Conversely, suppose (φ, M) is a regular submanifold of N. Let $p \in M$. Then for any neighborhood $U \subset M$ of p, there exists a neighborhood V of $q = \varphi(p)$ in N such that $\varphi(U) = \varphi(M) \cap V$. By Theorem 1.2.2, there exist local coordinate systems $(U_1; u^i)$ for p and $(V_1; v^\alpha)$ for q such that $\varphi(U_1) \subset V_1$, and $\varphi|_{U_1}$ can be expressed in local coordinates as

$$\varphi(u^1, \cdots, u^m) = (u^1, \cdots, u^m, 0, \cdots, 0).$$

We may assume that $U_1 \subset U$. Hence we can choose $V_1 \subset V$ with $\varphi(U_1) = \varphi(M) \cap V_1$. Here we can see that $\varphi(M) \cap V$ is actually defined by

$$v^{m+1} = \cdots = v^n = 0.$$

For each $q \in \varphi(M)$, use V_q to represent the corresponding neighborhood V_1 of q in N defined above. Let $W = \bigcup_{q\in\varphi(M)} V_q$. It is obvious that W is an open submanifold of N containing $\varphi(M)$. We only need to show that $\varphi(M)$ is relatively closed in W, or equivalently, $\overline{\varphi(M)} \cap W = \varphi(M)$. Choose any point $s \in \overline{\varphi(M)} \cap W$. Then there exists $q \in \varphi(M)$ such that $s \in V_q$. By the choice of V_q, $\varphi(M) \cap V_q$ is a relatively closed subset of V_q. Since $s \in \overline{\varphi(M)} \cap V_q$, we have $s \in \varphi(M) \cap V_q$. Therefore $\overline{\varphi(M)} \cap W \subset \varphi(M)$. This proves that (φ, M) is a closed submanifold of the open submanifold W of N. □

Theorem 1.2.4 Suppose (φ, M) is a submanifold of a smooth manifold N. If M is compact, then $\varphi : M \to N$ is a regular imbedding.

Proof. Because $\varphi : M \to \varphi(M) \subset N$ is a continuous bijection from a compact space to a Hausdorff space, it must be a closed map and then a homeomorphism. Therefore, (φ, M) is a regular submanifold of N by definition. □

2 Exterior Differential Calculus

2.1 Tensor Bundles and Vector Bundles

Suppose M is an m-dimensional smooth manifold, T_p and T_p^* are the tangent and cotangent space of M at p. Then there is an (r, s)-type tensor space

$$T_s^r(p) = \underbrace{T_p \otimes \cdots \otimes T_p}_{r \text{ terms}} \otimes \underbrace{T_p^* \otimes \cdots \otimes T_p^*}_{s \text{ terms}}$$

of M at p, which is an m^{r+s}-dimensional vector space. Let

$$T_s^r = \bigcup_{p\in M} T_s^r(p).$$

We will introduce a topology on T_s^r so that it becomes a Hausdorff space with a countable basis, and then define a smooth structure to make it a smooth manifold.

Suppose V is an m-dimensional vector space over $\mathbb{R}$. Choose a basis $\{e_1, e_2, \cdots, e_m\}$ in V, and then each element $y \in V$ can be expressed as a row coordinate vector

$$y = (y^1, y^2, \cdots, y^m).$$

The space V^r_s of all (r,s)-type tensors on V has a basis

$$e_{i_1}\otimes e_{i_2}\otimes\cdots e_{i_r}\otimes e^{*j_1}\otimes e^{*j_2}\otimes\cdots e^{*j_s},\quad 1\le i_\alpha, j_\beta\le m.$$

Thus the elements of V^r_s can also be expressed by components.

Consider a coordinate neighborhood U on M with local coordinates $u^1,\cdots,u^m$. Then for any $p\in U$,

$$\left(\frac{\partial}{\partial u^{i_1}}\right)_p\otimes\cdots\otimes\left(\frac{\partial}{\partial u^{i_r}}\right)_p\otimes(\mathrm{d}u^{j_1})_p\otimes\cdots\otimes(\mathrm{d}u^{j_s})_p,\quad 1\le i_\alpha, j_\beta\le m$$

forms a basis of $T^r_s(p)$. We can define a map

$$\varphi_U : U\times V^r_s\to\bigcup_{p\in U}T^r_s(p)$$

such that for any $p\in U, 1\le i_\alpha, j_\beta\le m$, we have

$$\begin{aligned}&\varphi_U(p, e_{i_1}\otimes\cdots e_{i_r}\otimes e^{*j_1}\otimes\cdots e^{*j_s})\\ =&\left(\frac{\partial}{\partial u^{i_1}}\right)_p\otimes\cdots\otimes\left(\frac{\partial}{\partial u^{i_r}}\right)_p\otimes(\mathrm{d}u^{j_1})_p\otimes\cdots\otimes(\mathrm{d}u^{j_s})_p\in T^r_s(p).\end{aligned}$$

Such a φ_U is a one-to-one correspondence.

Choose a coordinate covering $\{U_1, U_2,\cdots\}$ of M, with corresponding maps $\{\varphi_1,\varphi_2,\cdots\}$. Let the set of images of all open subsets of $U_i\times V^r_s$ under the map φ_i be a topological basis for T^r_s. Such a topology makes T^r_s into a Hausdorff space with a countable basis, and each map φ_i is then a homeomorphism.

Fix a point $p\in U$. The map $\varphi_{U,p} : V^r_s\to T^r_s(p)$ defined by

$$\varphi_{U,p}(y)=\varphi_U(p,y),\quad y\in V^r_s$$

is a linear isomorphism. If W is another coordinate neighborhood of M containing p, let

$$g_{UW}(p)=\varphi^{-1}_{W,p}\circ\varphi_{U,p} : V^r_s\to V^r_s.$$

Then obviously $g_{UW}(p)\in\mathrm{GL}(V^r_s)$. Therefore, for any two coordinate neighborhoods U, W of M with $U\cap W\ne\varnothing$, the map

$$g_{UW} : U\cap W\to\mathrm{GL}(V^r_s)$$

is well-defined. Moreover, it can be shown that g_{UW} is actually some tensor products of the Jacobian matrix of the change of local coordinates, thus g_{UW} is smooth on $U\cap W$.

Now we construct the smooth structure of T^r_s. First,

$$\{\varphi_1(U_1 \times V^r_s), \varphi_2(U_2 \times V^r_s), \cdots\}$$

forms an open covering of T^r_s. The coordinates of a point $\varphi_i(p, y)$ in the coordinate neighborhood $\varphi_i(U_i \times V^r_s)$ are

$$(u^\alpha_i(p), y^{i_1\cdots i_r}_{j_1\cdots j_s}),$$

where u^α_i is a local coordinate in the coordinate neighborhood U_i of the manifold M, and $y^{i_1\cdots i_r}_{j_1\cdots j_s}$ is the component of $y \in V^r_s$ with respect to the basis $e_{i_1} \otimes \cdots e_{i_r} \otimes e^{*j_1} \otimes \cdots e^{*j_s}$ of V^r_s. Noting that for $U_i \cap U_j \neq \varnothing$, $g_{ij} : U_i \cap U_j \to \mathrm{GL}(V^r_s)$ is smooth, we see that the coordinate covering of T^r_s given above is C^∞-compatible. Thus T^r_s becomes a smooth manifold. Obviously, the natural projection

$$\pi : T^r_s \to M,$$

which maps each element in $T^r_s(p)$ to the point $p \in M$, is a smooth surjection. The smooth manifold T^r_s is called a **type (r, s)-tensor bundle** on M, π is called the **bundle projection**, and $T^r_s(p)$ is called the **fiber** of the bundle T^r_s at p.

Letting $r = 1, s = 0$, we get the **tangent bundle** of M, denoted by $T(M)$. Letting $r = 0, s = 1$, we get the **cotangent bundle** of M, denoted by $T^*(M)$. Replacing $T^r(p)$ by $\Lambda^r(T_p)$ and V^r by $\Lambda^r(V)$, and following the above procedure, we can construct **exterior vector bundles**

$$\Lambda^r(M) = \bigcup_{p\in M} \Lambda^r(T_p)$$

on M. Similarly, we can also construct **exterior form bundles**

$$\Lambda^r(M^*) = \bigcup_{p\in M} \Lambda^r(T^*_p)$$

on M.

Suppose $f : M \to T^r_s$ is a smooth map such that $\pi \circ f = \mathrm{id}_M$, i.e., $f(p) \in T^r_s(p)$ for any $p \in M$, then f is called a **smooth section** of the tensor bundle T^r_s, or a **type (r, s)-smooth tensor field** on M. A section of a tangent bundle is a **tangent vector field** on M, and a section of a cotangent bundle is a **differential 1-form**. A smooth section of the exterior form bundle $\Lambda^r(M*)$ is called an **exterior differential form** of degree r on M.

Definition 2.1.1 Suppose E, M are two smooth manifolds, and $\pi : E \to M$ is a smooth surjection. Let V be a q-dimensional vector space. If an open covering $\{U_1, U_2, \cdots\}$ of M and a set of maps $\{\varphi_1, \varphi_2, \cdots\}$ satisfy the following conditions:

1. Every map φ_i is a diffeomorphism from $U_i \times V$ to $\pi^{-1}(U_i)$, and for any $p \in U_i, y \in V$,
$$\pi \circ \varphi_i(p, y) = p.$$

2. For any fixed $p \in U_i$, let
$$\varphi_{i,p}(y) = \varphi_i(p, y), \quad y \in V.$$
Then $\varphi_{i,p} : V \to \pi^{-1}(p)$ is a homeomorphism. When $U_i \cap U_j \neq \varnothing$, for any $p \in U_i \cap U_j$,
$$g_{ij}(p) = \varphi_{j,p}^{-1} \circ \varphi_{i,p} : V \to V$$
is a linear automorphism of V, i.e. $g_{ij}(p) \in \mathrm{GL}(V)$.

3. When $U_i \cap U_j \neq \varnothing$, the map $g_{ij} : U_i \cap U_j \to \mathrm{GL}(V)$ is smooth.

then (E, M, π) is called a (real) q-dimensional **vector bundle** on M, where E is called the **bundle space**, M is called the **base space**, π is called the **bundle projection**, and V is called the **typical fiber**.

For any $p \in M$, define $E_p = \pi^{-1}(p)$ and call it the **fiber** of the vector bundle E at the point p. For a coordinate neighborhood U_i of M containing p, the linear structure of the typical fiber V can be transported to E_p through the map $\varphi_{i,p}$. Condition 2 ensures that the linear structure of E_p is independent of the choice of U_i and φ_i.

The product manifold $M \times V$ is the most simple example of a vector bundle, called the **trivial bundle** over M, or the **product bundle**.

The map $g_{ij} : U_i \cap U_j \to \mathrm{GL}(V)$ satisfies the following compatibility conditions:

1. for $p \in U_i$, $g_{ii}(p) = \mathrm{id}_V$;

2. if $p \in U_i \cap U_j \cap U_k \neq \varnothing$, then $g_{ki}(p) \circ g_{jk}(p) \circ g_{ij}(p) = \mathrm{id}_V$.

The set $\{g_{ij}\}$ is called the family of **transition functions** of the vector bundle (E, M, π).

Theorem 2.1.1 Suppose M is an m-dimensional smooth manifold, $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ is an open covering of M, and V is a q-dimensional vector space. If for any pair of indices $\alpha, \beta \in \mathcal{A}$ where $U_\alpha \cap U_\beta \neq \varnothing$, there exists a smooth map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(V)$ that satisfies compatibility conditions, then there exists a q-dimensional vector bundle (E, M, π) which has $\{g_{\alpha\beta}\}$ as its transition functions.

For a vector bundle (E, M, π) with V as its typical fiber, we can construct another vector bundle $(E^*, M, \tilde{\pi})$ with V^* as its typical fiber, whose transition functions are the dual maps of the transition functions of (E, M, π). The vector bundle E^* is called the **dual bundle** of E. In fact, the cotangent bundle is exactly the dual bundle of the tangent bundle. Similarly, we can construct the **direct sum** and the **tensor product** of vector bundles.

Definition 2.1.2 Suppose $s : M \to E$ is a smooth map. If $\pi \circ s = \mathrm{id}_M$, then s is called a **smooth section** of the vector bundle (E, M, π). The set of all smooth sections of the vector bundle (E, M, π) is denoted by $\Gamma(E)$.

Suppose $s, s_1, s_2 \in \Gamma(E)$ and $\alpha \in C^\infty(M)$. For any $p \in M$, let

$$(s_1 + s_2)(p) = s_1(p) + s_2(p),$$
$$(\alpha s)(p) = \alpha(p)s(p).$$

Then $s_1 + s_2$ and αs are also smooth sections of the vector bundle E. This makes $\Gamma(E)$ into a $C^\infty(M)$-module.

2.2 Exterior Differentiation

Suppose M is an m-dimensional smooth manifold. Let

$$A^r(M) = \Gamma(\Lambda^r(M^*))$$

be the space of the smooth sections of the exterior form bundle $\Lambda^r(M^*)$. The elements of $A^r(M)$ are called **exterior differential r-forms** on M. Similarly, let

$$A(M) = \Gamma(\Lambda(M^*))$$

be the space of all the smooth sections of the vector bundle $\Lambda(M^*)$. The elements of $A(M)$ are called **exterior differential forms** on M. $A(M)$ has the expression as the direct sum

$$A(M) = \sum_{r=0}^{m} A^r(M).$$

The wedge product $\wedge$ defines a map

$$\wedge : A^r(M) \times A^s(M) \to A^{r+s}(M)$$

for each r, s which makes $A(M)$ into a **graded algebra**.

Lemma 2.2.1 Suppose (U, φ) is a coordinate chart in a smooth manifold M, $V \neq \varnothing$ is an open set in M with $\overline{V}$ compact, and $\overline{V} \subset U$. Then there exists a smooth function $h : M \to \mathbb{R}$ such that

1. $0 \leq h \leq 1$;
2. $h(p) = \begin{cases} 1, & p \in V; \\ 0, & p \notin U. \end{cases}$

Theorem 2.2.2 Suppose M is an m-dimensional smooth manifold. Then there exists a unique map

$$\mathrm{d} : A(M) \to A(M)$$

such that $\mathrm{d}(A^r(M)) \subset A^{r+1}(M)$ and such that it satisfies the following properties:

1. For any $\omega_1, \omega_2 \in A(M)$, $\mathrm{d}(\omega_1 + \omega_2) = \mathrm{d}\omega_1 + \mathrm{d}\omega_2$.
2. Suppose $\omega_1 \in A^r(M)$, then for any $\omega_2 \in A(M)$,
$$\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^r \omega_1 \wedge \mathrm{d}\omega_2.$$
3. If f is a smooth function on M, i.e. $f \in A^0(M)$, then $\mathrm{d}f$ is precisely the differential of f.
4. If $f \in A^0(M)$, then $\mathrm{d}(\mathrm{d}f) = 0$.

The map d defined above is called the **exterior derivative**.

Proof. First we show that id the exterior operator d exists, then it is a local operator. It suffices to show that $\omega|_U = 0$ implies $(\mathrm{d}\omega)|_U = 0$. Choose any point $p \in U$. Then there is an open neighborhood W containing p such that $p \in W \subset \overline{W} \subset U$. By Lemma 2.2.1, there exists a smooth function h on M such that

$$h(p') = \begin{cases} 1, & p \in W; \\ 0, & p \notin U. \end{cases}$$

Thus $h\omega \in A(M)$ and $h\omega = 0$. Therefore

$$\mathrm{d}h \wedge \omega + h\mathrm{d}\omega = 0,$$

and hence $(\mathrm{d}\omega)|_W = 0$. The arbitrarity of p then implies that the restriction of $\mathrm{d}\omega$ in U must be zero.

Suppose ω is an exterior differential form defined on the open set U. Using Lemma 2.2.1, for any point $p \in U$, there is a coordinate neighborhood $U_1 \subset U$ of p and an exterior differential form $\tilde{\omega}$ defined on M such that $\tilde{\omega}|_{U_1} = \omega|_{U_1}$. Thus we can define $\mathrm{d}\tilde{\omega}|_{U_1} = \mathrm{d}\omega|_{U_1}$. Since d is a local operator, the above definition is independent of the choice of $\tilde{\omega}$. $\mathrm{d}\omega$ is therefore well-defined.

Now we show the uniqueness of the exterior derivative d within a local coordinate neighborhood. We only need to show this for a monomial. Suppose in a coordinate neighborhood U, ω is expressed by

$$\omega = a\mathrm{d}u^1 \wedge \cdots \wedge \mathrm{d}u^r,$$

where a is a smooth function on U. By the properties of d, we see that

$$\mathrm{d}\omega = \mathrm{d}a \wedge \mathrm{d}u^1 \wedge \cdots \wedge \mathrm{d}u^r,$$

where $\mathrm{d}a$ is the differential of the function a. Thus $\mathrm{d}\omega$ restricted to the coordinate neighborhood U has a completely determined form.

Suppose

$$\omega|_U = a_{i_1 \cdots i_r} \mathrm{d}u^{i_1} \wedge \cdots \wedge \mathrm{d}u^{i_r}.$$

Then we can define

$$\mathrm{d}(\omega|_U) = \mathrm{d}a_{i_1 \cdots i_r} \wedge \mathrm{d}u^{i_1} \wedge \cdots \wedge \mathrm{d}u^{i_r}.$$

Obviously, $\mathrm{d}(\omega|_U)$ is an exterior differential $(r+1)$-form on U satisfying conditions 1 and 3. To show that 2 holds, we need only consider any two monomials

$$\alpha_1 = a\mathrm{d}u^{i_1} \wedge \cdots \wedge \mathrm{d}u^{i_r}$$
$$\alpha_2 = b\mathrm{d}u^{j_1} \wedge \cdots \wedge \mathrm{d}u^{j_r}.$$

By the definition, we have

$$\begin{aligned}
\mathrm{d}(\alpha_1 \wedge \alpha_2) &= \mathrm{d}(ab) \wedge \mathrm{d}u^{i_1} \wedge \cdots \wedge \mathrm{d}u^{i_r} \wedge \mathrm{d}u^{j_1} \wedge \cdots \wedge \mathrm{d}u^{j_r} \\
&= (a\mathrm{d}b + b\mathrm{d}a) \wedge \mathrm{d}u^{i_1} \wedge \cdots \wedge \mathrm{d}u^{i_r} \wedge \mathrm{d}u^{j_1} \wedge \cdots \wedge \mathrm{d}u^{j_r} \\
&= (\mathrm{d}a \wedge \mathrm{d}u^{i_1} \wedge \cdots \wedge \mathrm{d}u^{i_r}) \wedge (b\mathrm{d}u^{j_1} \wedge \cdots \wedge \mathrm{d}u^{j_r}) \\
&\quad + (-1)^r (a\mathrm{d}u^{i_1} \wedge \cdots \wedge \mathrm{d}u^{i_r}) \wedge (\mathrm{d}b \wedge \mathrm{d}u^{j_1} \wedge \cdots \wedge \mathrm{d}u^{j_r}) \\
&= \mathrm{d}\alpha_1 \wedge \alpha_2 + (-1)^r \alpha_1 \wedge \mathrm{d}\alpha_2.
\end{aligned}$$

Property 2 is therefore established.

We now prove condition 4. Suppose f is a smooth function on M. Then on U it satisfies

$$\mathrm{d}f = \frac{\partial f}{\partial u^i}\mathrm{d}u^i.$$

Since f is C^∞, its higher then first order partial derivatives are independent of the order taken, i.e.,

$$\frac{\partial^2 f}{\partial u^i \partial u^j} = \frac{\partial^2 f}{\partial u^j \partial u^i}.$$

Therefore

$$\begin{aligned}
\mathrm{d}(\mathrm{d}f) &= \mathrm{d}\left(\frac{\partial f}{\partial u^i}\right) \wedge \mathrm{d}u^i \\
&= \frac{\partial^2 f}{\partial u^i \partial u^j}\mathrm{d}u^j \wedge \mathrm{d}u^i \\
&= \frac{1}{2}\left(\frac{\partial^2 f}{\partial u^i \partial u^j} - \frac{\partial^2 f}{\partial u^j \partial u^i}\right)\mathrm{d}u^j \wedge \mathrm{d}u^i \\
&= 0.
\end{aligned}$$

If W is another coordinate neighborhood, we obtain by the local property of the exterior derivative operator and its uniqueness in a local coordinate neighborhood that

$$(\mathrm{d}(\omega|_U))|_{U\cap W} = \mathrm{d}(\omega|_{U\cap W}) = (\mathrm{d}(\omega|_W))|_{U\cap W}.$$

Hence the exterior derivative operator d is uniformly defined above on $U\cap W$, i.e. d is an operator defined on M globally. This proves the existence of the operator d satisfying the conditions of the theorem. □

Theorem 2.2.3 (Poincare's Lemma) For any exterior differential form ω, $\mathrm{d}(\mathrm{d}\omega) = 0$.

Proof. Since d is a linear operator, we need only prove the lemma when ω is a monomial. By the local properties of d, it suffices to assume that

$$\omega = a\mathrm{d}u^1 \wedge \cdots \wedge \mathrm{d}u^r.$$

Hence

$$\mathrm{d}\omega = \mathrm{d}a \wedge \mathrm{d}u^1 \wedge \cdots \mathrm{d}u^r.$$

Differentiating one more time and applying conditions 2 and 4, we have

$$\begin{aligned} d(d\omega) &= d(da) \wedge du^1 \wedge \cdots \wedge du^r \\ &\quad - da \wedge d(du^1) \wedge \cdots \wedge du^r + \cdots \\ &= 0. \end{aligned}$$

□

Suppose $f : M \to N$ is a smooth map from a smooth manifold M to a smooth manifold N. Then f induces a tangent mapping $f_* : T_p(M) \to T_{f(p)}(N)$ at every point $p \in M$. For $\omega \in A^0(N)$, define

$$f^*\omega = \omega \circ f \in A^0(M).$$

For $\omega \in A^r(N), r \geq 1$, let $f^*\omega$ be an element of $A^r(M)$ such that for any r smooth tangent vector fields $X_1, X_2, \cdots, X_r$ on M,

$$\langle X_1 \wedge X_2 \wedge \cdots \wedge X_r, f^*\omega \rangle_p = \langle f_*X_1 \wedge f_*X_2 \wedge \cdots \wedge f_*X_r, \omega \rangle_{f(p)}, \quad p \in M,$$

where $\langle \cdot, \cdot \rangle$ can be computed by

$$\left\langle \frac{\partial}{\partial u^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial u^{i_r}}, du^{j_1} \wedge \cdots \wedge du^{j_r} \right\rangle_p = \delta^{j_1 \cdots j_r}_{i_1 \cdots i_r}.$$

Under this definition, the map f^* distributes over the wedge product, i.e.

$$f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta, \quad \omega, \eta \in A(N).$$

Theorem 2.2.4 Suppose M, N are smooth manifold and $f : M \to N$ is a smooth map. Then the following diagram commutes:

$$\begin{array}{ccc} A(N) & \xrightarrow[d]{} & A(N) \\ \Big\downarrow{\scriptstyle f^*} & & {\scriptstyle f^*}\Big\downarrow \\ A(M) & \xrightarrow{d} & A(M) \end{array}$$

Proof. We can prove the equation $f^*(d\omega) = d(f^*\omega)$ for monomials ω by induction on its degree. □

2.3 Integrals of Differential Forms

Definition 2.3.1 An m-dimensional smooth manifold M is called **orientable** if there exists a continuous and nonvanishing exterior differential m-form ω on M. If M is given such an ω, then M is said to be **oriented**. If two such forms are given on M such that they differ by a function factor which is always positive, then we say that they assign the same **orientation** to M.

If ω, η are two exterior differential m-forms giving orientations to M, then there exists a nonvanishing continuous function f such that $\eta = f\omega$. When M is connected, f retains the same sign on the whole M. Therefore the orientation given by η is either identical to the one given by ω or the one given by $-\omega$. This implies that there exist exactly two orientations on a connected orientable manifold.

Suppose M is oriented by the exterior differential form ω, and $(U; u^i)$ is any local coordinate system on M. Then $\mathrm{d}u^1 \wedge \cdots \wedge \mathrm{d}u^m$ and $\omega|_U$ are the same up to a nonzero factor. If the factor is positive, then $(U; u^i)$ is said to be a coordinate system **consistent** with the orientation of M.

Definition 2.3.2 Suppose $f : M \to \mathbb{R}$ is a real function on M. The **support** of f is the closure of the set of points at which f is nonzero, i.e.

$$\operatorname{supp} f = \overline{\{p \in M \mid f(p) \neq 0\}}.$$

If ϕ is an exterior differential form, the the support of ϕ is

$$\operatorname{supp} \phi = \overline{\{p \in M \mid \phi(p) \neq 0\}}.$$

Definition 2.3.3 Suppose Σ_0 is an open covering of M. If every compact subset of M intersects only finitely many elements of Σ_0, then Σ_0 is called a **locally finite** open covering of M.

Theorem 2.3.1 Suppose Σ is a topological basis of the manifold M. Then there is a subset Σ_0 of Σ such that Σ_0 is a locally finite open covering of M.

Proof. The second countability of M suggests that there exists a countable open covering $\{U_i\}$ of M such that the closure $\overline{U}_i$ of every U_i is compact. Let

$$P_i = \bigcup_{r=1}^{i} \overline{U}_r, \quad i = 1, 2, \cdots,$$

then P_i is compact, $P_i \subset P_{i+1}$ and

$$\bigcup_{i=1}^{\infty} P_i = M.$$

Now we inductively construct another sequence of compact sets Q_i satisfying $P_i \subset Q_i \subset \mathring{Q}_{i+1}$ for each i. Let $Q_0 = \varnothing$. Assuming that $Q_0, \cdots, Q_{i-1}$ have been constructed, we are going to construct Q_i. Since $Q_{i-1} \cup P_i$ is compact, there exist finitely many elements $U_\alpha, 1 \leq \alpha \leq s$ of $\{U_i\}$ such that

$$Q_{i-1} \cup P_i \subset \bigcup_{\alpha=1}^{s} U_\alpha.$$

Let

$$Q_i = \bigcup_{\alpha=1}^{s} \overline{U}_\alpha,$$

then Q_i satisfies $P_{i-1} \subset Q_{i-1} \subset \mathring{Q}_i$ and $P_i \subset Q_i$. Obviously we also have

$$\bigcup_{i=1}^{\infty} Q_i = M.$$

Denote $Q_{-1} = \varnothing$ and let

$$L_i = Q_i - \mathring{Q}_{i-1}, \quad K_i = \mathring{Q}_{i+1} - Q_{i-2}$$

for each positive integer i. Then L_i is compact, K_i is open, and $L_i \subset K_i$. Since Σ is a topological basis of M, K_i can be expressed as a union of elements of Σ. These elements form an open covering of L_i, and hence there exist finitely many elements $V_{i,\alpha}, 1 \leq \alpha \leq r_i$ in Σ such that

$$L_i \subset \bigcup_{\alpha=1}^{r_i} V_{i,\alpha} \subset K_i$$

for each i. Because

$$\bigcup_{i=1}^{\infty} L_i = \bigcup_{i=1}^{\infty} Q_i = M,$$

we see that

$$\Sigma_0 = \{V_{i,\alpha}, 1 \leq \alpha \leq r_i, i \geq 1\}$$

is a subcovering of Σ.

To show the local finiteness, we consider an arbitrary compact set A. There exists a sufficiently large integer i such that $A \subset P_i \subset Q_i$. For $j \geq i+2$,

$$K_j = \mathring{Q}_{j+1} - Q_{j-2} \subset \mathring{Q}_{j+1} - Q_i,$$

thus

$$A \cap V_{j,\alpha} \subset Q_i \cap K_j = \varnothing, \quad 1 \leq \alpha \leq r_j.$$

Therefore only finitely many elements of Σ_0 intersect A. □

Theorem 2.3.2 (Partition of Unity Theorem) Suppose Σ is an open covering of a smooth manifold M. Then there exists a family of smooth functions $\{g_\alpha\}$ on M satisfying the following conditions:

1. $0 \leq g_\alpha \leq 1$, and $\operatorname{supp} g_\alpha$ is compact for each α. Moreover, there exists an open set $W_i \subset \Sigma$ such that $\operatorname{supp} g_\alpha \subset W_i$;
2. For each point $p \in M$, there is a neighborhood U of p that intersects $\operatorname{supp} g_\alpha$ for only finitely many α;
3. $\sum_\alpha g_\alpha = 1$.

The family $\{g_\alpha\}$ is called a **partition of unity** subordinate to the open covering Σ.

Proof. Because M is a manifold, there is a topological basis $\Sigma_0 = \{U_\alpha\}$ such that each U_α is a coordinate neighborhood, $\overline{U}_\alpha$ is compact, and there exists $W_i \in \Sigma$ such that $\overline{U}_\alpha \subset W_i$. By Theorem 2.3.1, we may assume that Σ_0 itself is a locally finite open covering of M with countably many elements.

For each U_α, we construct V_α by a contraction of U_α such that $\overline{V}_\alpha \subset U_\alpha$ and $\{V_\alpha\}$ is also an open covering for M. Let

$$W_\alpha = \bigcup_{i \neq \alpha} U_i.$$

Then $M - W_\alpha$ is a closed set contained in U_α and hence $\overline{U}_\alpha$. The compactness of $\overline{U}_\alpha$ implies that $M - W_\alpha$ is also compact. Thus there are finitely many coordinate neighborhoods $W_{\alpha,s}, 1 \leq s \leq r_\alpha$ such that $\overline{W}_{\alpha,s} \subset U_\alpha$ and

$$M - W_\alpha \subset \bigcup_{s=1}^{r_\alpha} W_{\alpha,s}.$$

Now let

$$V_\alpha = \bigcup_{s=1}^{r_\alpha} W_{\alpha,s},$$

then the V_α are as desired.

By Lemma 2.2.1, there exist smooth functions h_α with $0 \leq h_\alpha \leq 1$ on M such that

$$h_\alpha(p) = \begin{cases} 1, & p \in V_\alpha; \\ 0, & p \notin U_\alpha. \end{cases}$$

Then $\operatorname{supp} h_\alpha \subset \overline{U}_\alpha$. For any point $p \in M$, there exists a neighborhood U such that $\overline{U}$ is compact. The local finiteness of Σ_0 implies that $\overline{U}$ intersects only finitely many elements of Σ_0, and there are only finitely many nonzero terms in the summation $\sum_\alpha h_\alpha(p)$. Thus $h = \sum_\alpha h_\alpha$ defines a smooth function on M. Since $\{V_\alpha\}$ covers M, any point $p \in M$ must lie in some V_α, and thus $h(p) \geq h_\alpha(p) = 1$. Let $g_\alpha = h_\alpha / h$, then the family $\{g_\alpha\}$ satisfies all the conditions of the theorem. □

Suppose M is an m-dimensional smooth manifold, and φ is an exterior differential m-form on M with a compact support. Choose any coordinate covering $\Sigma = \{W_i\}$ which is consistent with the orientation of M, and suppose that $\{g_\alpha\}$ is a partition of unity subordinate to Σ. Then $\varphi = \sum_\alpha (g_\alpha \cdot \varphi)$ and $\operatorname{supp}(g_\alpha \cdot \varphi)$ is contained in some coordinate neighborhood $W_i \in \Sigma$. Suppose $u^1, \cdots, u^m$ is a coordinate system of W_i, with respect to which $g_\alpha \cdot \varphi$ has the expression as

$$f(u^1, \cdots, u^m) \mathrm{d}u^1 \wedge \cdots \wedge \mathrm{d}u^m.$$

The integral of $g_\alpha \cdot \varphi$ is then defined to be

$$\int_M g_\alpha \cdot \varphi = \int_{W_i} g_\alpha \cdot \varphi = \int_{W_i} f(u^1, \cdots, u^m) \mathrm{d}u^1 \cdots \mathrm{d}u^m,$$

where the right hand side is the usual Riemann integral.

We need to show that the right hand side is independent of the choice of the coordinate system $(W_i; u^1, \cdots, u^m)$. Suppose $\operatorname{supp}(g_\alpha \cdot \varphi) \subset W_i \cap W_j$, where W_i, W_j have the local coordinates u^k, v^k consistent with the orientation of M, respectively. The the Jacobian satisfies

$$J = \frac{\partial(v^1, \cdots, v^m)}{\partial(u^1, \cdots, u^m)} > 0.$$

Suppose $g_\alpha \cdot \varphi$ is expressed in W_i and W_j, respectively, by

$$\begin{aligned} g_\alpha \cdot \varphi &= f \mathrm{d}u^1 \wedge \cdots \wedge \mathrm{d}u^m \\ &= \tilde{f} \mathrm{d}v^1 \wedge \cdots \wedge \mathrm{d}v^m. \end{aligned}$$

Then we have

$$f = \tilde{f} \cdot J = \tilde{f} \cdot |J|,$$

and $\operatorname{supp} f = \operatorname{supp} \tilde{f} = \operatorname{supp}(g_\alpha \cdot \varphi) \subset W_i \cap W_j$. Therefore

$$\begin{aligned}
\int_{W_j} \tilde{f} \mathrm{d}v^1 \cdots \mathrm{d}v^m &= \int_{W_i \cap W_j} \tilde{f} \mathrm{d}v^1 \cdots \mathrm{d}v^m \\
&= \int_{W_i \cap W_j} \tilde{f} \cdot |J| \mathrm{d}u^1 \cdots \mathrm{d}u^m \\
&= \int_{W_i \cap W_j} f \mathrm{d}u^1 \cdots \mathrm{d}u^m \\
&= \int_{W_i} f \mathrm{d}u^1 \cdots \mathrm{d}u^m,
\end{aligned}$$

i.e. the integral of $g_\alpha \cdot \varphi$ on M is well-defined.

Since $\operatorname{supp} \varphi$ is compact, it only intersects finitely many $\operatorname{supp} g_\alpha$. Let

$$\int_M \varphi = \sum_\alpha \int_M g_\alpha \cdot \varphi.$$

Now we show that the right hand side is independent of the choice of the partition of unity $\{g_\alpha\}$. Suppose $\{\tilde{g}_\beta\}$ is another partition of unity subordinate to Σ. Then

$$\begin{aligned}
\sum_\beta \int_M \tilde{g}_\beta \cdot \varphi &= \sum_{\alpha,\beta} \int_M g_\alpha \cdot \tilde{g}_\beta \cdot \varphi \\
&= \sum_\alpha \int_M \sum_\beta \tilde{g}_\beta \cdot g_\alpha \cdot \varphi \\
&= \sum_\alpha \int_M g_\alpha \cdot \varphi.
\end{aligned}$$

In conclusion, the value of

$$\int_M \varphi$$

is well-defined, and is called the **integral** of the exterior differential form φ on M.

If φ is an exterior differential r-form, $r < m$, with compact support, then we can define the integral of φ on any r-dimensional submanifold N of M. Suppose $h : N \to M$ is an r-dimensional imbedding of N into M. Then $h^*\varphi$ is an exterior differential r-form on the r-dimensional smooth manifold N

with compact support. The integral of φ on the submanifold $h(N)$ of M is then defined as

$$\int_{h(N)} \varphi = \int_N h^*\varphi.$$

2.4 Stokes' Formula

Definition 2.4.1 Suppose M is an m-dimensional smooth manifold. A **region D with boundary** is a subset of M with two kinds of points:

1. Interior points, each of which has a neighborhood in M contained in D.

2. Boundary points, for each of which there is a coordinate system $(U; u^i)$ such that $u^i(p) = 0$ and

$$U \cap D = \{q \in U \mid u^m(q) \geq 0\}.$$

A coordinate system u^i with the above property is called an **adapted coordinate system** for the boundary point p. The set B of all the boundary points of D is called the **boundary** of D.

Theorem 2.4.1 The boundary B of a region D is a regular imbedded closed submanifold. Furthermore, if M is orientable, then B is also orientable.

Proof. The boundary B of the region D is a closed subset of M. Suppose $(U; u^i)$ is an adapted coordinate neighborhood, then

$$U \cap B = \{q \in U \mid u^m(q) = 0\}.$$

Thus B is a regular imbedded closed submanifold of M.

Now suppose M is an orientable manifold. Choose an adapted coordinate neighborhood $(U; u^i)$ which is consistent with the orientation of M at an arbitrary point $p \in B$. Then $(u^1, \cdots, u^{m-1})$ is a local coordinate system of B at the point p. Let

$$(-1)^m \mathrm{d}u^1 \wedge \cdots \wedge \mathrm{d}u^{m-1}$$

specify the orientation of the boundary B in the coordinate neighborhood $U \cap B$ of the point p.

Suppose $(V; v^i)$ is another adapted coordinate neighborhood of the boundary point p consistent with the orietation of M. Then

$$\frac{\partial(v^1, \cdots, v^m)}{\partial(u^1, \cdots, u^m)} > 0.$$

Moreover, the sign of v^m is the same as that of u^m, and $v^m = 0$ holds whenever $u^m = 0$. This means that

$$\left.\frac{\partial v^m}{\partial u^i}\right|_q = 0, \quad 1 \le i \le m-1,$$

and that

$$\left.\frac{\partial v^m}{\partial u^m}\right|_q > 0$$

for any $q \in U \cap V \cap B$. Therefore

$$\frac{\partial(v^1, \cdots, v^{m-1})}{\partial(u^1, \cdots, u^{m-1})} > 0$$

holds within $U \cap V \cap B$. This shows that $(-1)^m \mathrm{d}u^1 \wedge \cdots \wedge \mathrm{d}u^{m-1}$ and $(-1)^m \mathrm{d}v^1 \wedge \cdots \wedge \mathrm{d}v^{m-1}$ give consistent orientations in $U \cap V \cap B$. Therefore, the orientation given by $(-1)^m \mathrm{d}u^1 \wedge \cdots \wedge \mathrm{d}u^{m-1}$ in $U \cap B$ can be extended to the whole boundary B. Hence B is orientable. □

The orientation of B given in the proof is called the **induced orientation** on the boundary B by an oriented manifold M. If D has the same orientation as M, we denote the boundary B with the induced orientation by ∂D.

Theorem 2.4.2 (Stokes' Formula) Suppose D is a region with boundary in an m-dimensional oriented manifold M, and ω is an exterior differential $(m-1)$-form on M with compact support. Then

$$\int_D \mathrm{d}\omega = \int_{\partial D} \omega.$$

If $\partial D = \varnothing$, then the integral on the right hand side is zero.

Proof. Suppose $\{U_i\}$ is a coordinate covering consistent with the orientation of M, and $\{g_\alpha\}$ is a subordinate partition of unity. Then

$$\omega = \sum_\alpha g_\alpha \cdot \omega.$$

The right hand side is a sum of finitely many terms since $\operatorname{supp}\omega$ is compact. Therefore

$$\int_D \mathrm{d}\omega = \sum_\alpha \int_D \mathrm{d}(g_\alpha \cdot \omega),$$

and

$$\int_{\partial D} \omega = \sum_\alpha \int_{\partial D} g_\alpha \cdot \omega.$$

Thus we may assume that $\operatorname{supp}\omega$ is contained in a coordinate neighborhood $(U; u^i)$ consistent with the orientation of M.

Suppose ω can be expressed as

$$\omega = \sum_{j=1}^{m} (-1)^{j-1} a_j \mathrm{d}u^1 \wedge \cdots \wedge \widehat{\mathrm{d}u^i} \wedge \cdots \wedge \mathrm{d}u^m,$$

where the a_j are smooth functions on U. Then

$$\mathrm{d}\omega = \left(\sum_{j=1}^{m} \frac{\partial a_j}{\partial u^j} \right) \mathrm{d}u^1 \wedge \cdots \wedge \mathrm{d}u^m.$$

Case 1: If $U \cap \partial D = \varnothing$, then

$$\int_{\partial D} \omega = 0.$$

Then either $U \subset M - D$ or U is contained in the interior of D. We only need to consider the latter one. Consider a cube

$$C = \{u \in \mathbb{R}^m \mid |u^i| \leq K, 1 \leq i \leq m\}$$

such that the image of U under coordinate maps is contained in the interior of C. The functions a_j can be smoothly extended to C by letting them be zero outside U. Noting that

$$\begin{aligned}\int_{-K}^{K} \frac{\partial a_j}{\partial u^j} \mathrm{d}u^j &= a_j(u^1, \cdots, u^{j-1}, K, u^{j+1}, \cdots, u^m) \\ &\quad - a_j(u^1, \cdots, u^{j-1}, -K, u^{j+1}, \cdots, u^m) \\ &= 0,\end{aligned}$$

we have

$$\begin{aligned}\int_U \frac{\partial a_j}{\partial u^j}\mathrm{d}u^1\cdots\mathrm{d}u^m &= \int_C \frac{\partial a_j}{\partial u^j}\mathrm{d}u^1\cdots\mathrm{d}u^m \\ &= \int_{|u^i|\le K, i\neq j}\left(\int_{-K}^{K}\frac{\partial a_j}{\partial u^j}\mathrm{d}u^j\right)\mathrm{d}u^1\cdots\widehat{\mathrm{d}u^j}\cdots\mathrm{d}u^m \\ &= 0\end{aligned}$$

for each j, and hence

$$\int_D \mathrm{d}\omega = \int_U\left(\sum_{j=1}^m \frac{\partial a_j}{\partial u^j}\right)\mathrm{d}u^1\cdots\mathrm{d}u^m = 0.$$

Case 2: If $U\cap\partial D\neq\varnothing$, we may assume that U is an adapted coordinate neighborhood consistent with the orientation of M. Then

$$U\cap D = \{q\in U \mid u^m(q)\ge 0\}$$

and

$$U\cap\partial D = \{q\in U \mid u^m(q) = 0\}.$$

Consider the cube

$$C = \{u\in\mathbb{R}^m \mid u^m\ge 0, |u^i|\le K, 1\le i\le m\}$$

such that the image of $U\cap D$ under coordinate maps is contained in the union of the interior of C and the boundary $u^m=0$. Noting that $\mathrm{d}u^m = 0$ on $U\cap\partial D$, we have

$$\begin{aligned}\int_{\partial D}\omega &= \int_{U\cap\partial D}\omega \\ &= \sum_{j=1}^m(-1)^{j-1}\int_{U\cap\partial D} a_j\mathrm{d}u^1\wedge\cdots\wedge\widehat{\mathrm{d}u^i}\wedge\cdots\wedge\mathrm{d}u^m \\ &= (-1)^{m-1}\int_{U\cap\partial D} a_m\mathrm{d}u^1\wedge\cdots\wedge\mathrm{d}u^{m-1} \\ &= -\int_{|u^i|\ge K, 1\le i<m} a_m(u^1,\cdots,u^{m-1},0)\mathrm{d}u^1\cdots\mathrm{d}u^{m-1}.\end{aligned}$$

On the other hand, since

$$\begin{aligned}\int_{U\cap D}\frac{\partial a_j}{\partial u^j}\mathrm{d}u^1\wedge\cdots\wedge\mathrm{d}u^m &= \int_{\substack{|u^i|\le K, i<m, i\neq j\\ 0\le u^m\le K}}\left(\int_{-K}^{K}\frac{\partial a_j}{\partial u^j}\mathrm{d}u^j\right)\mathrm{d}u^1\cdots\widehat{\mathrm{d}u^j}\cdots\mathrm{d}u^m \\ &= 0\end{aligned}$$

for $1 \leq j \leq m-1$, we have

$$\begin{aligned}
\int_D \mathrm{d}\omega &= \int_{U\cap D} \mathrm{d}\omega \\
&= \sum_{j=1}^{m} \int_{U\cap D} \frac{\partial a_j}{\partial u^j} \mathrm{d}u^1 \wedge \cdots \wedge \mathrm{d}u^m \\
&= \int_{U\cap D} \frac{\partial a_m}{\partial u^m} \mathrm{d}u^1 \wedge \cdots \wedge \mathrm{d}u^m \\
&= \int_{|u^i|\geq K, 1\leq i<m} \left(\int_0^K \frac{\partial a_m}{\partial u^m} \mathrm{d}u^m \right) \mathrm{d}u^1 \cdots \mathrm{d}u^{m-1} \\
&= \int_{|u^i|\geq K, 1\leq i<m} [a_m(u^1, \cdots, u^{m-1}, K) - a_m(u^1, \cdots, u^{m-1}, 0)] \mathrm{d}u^1 \cdots \mathrm{d}u^{m-1} \\
&= -\int_{|u^i|\geq K, 1\leq i<m} a_m(u^1, \cdots, u^{m-1}, 0) \mathrm{d}u^1 \cdots \mathrm{d}u^{m-1}.
\end{aligned}$$

In conclusion, we have

$$\int_D \mathrm{d}\omega = \int_{\partial D} \omega,$$

and the theorem is proved. □

We can view $A^r(M)$ as a cochain group with $\mathrm{d} : A^r(M) \to A^{r+1}(M)$ being the coboundary operator. Denote

$$Z^r(M, \mathbb{R}) = \{\omega \in A^r(M) \mid \mathrm{d}\omega = 0\}$$

and

$$B^r(M, \mathbb{R}) = \{\omega \in A^r(M) \mid \omega = \mathrm{d}\eta \text{ for some } \eta \in A^{r-1}(M)\}.$$

The elements of $Z^r(M, \mathbb{R})$ are called **closed** differential forms and the elements of $B^r(M, \mathbb{R})$ are called **exact** differential forms. Poincare's Lemma thus implies that $B^r(M, \mathbb{R}) \subset Z^r(M, \mathbb{R})$.

Definition 2.4.2 The quotient space

$$H^r(M, \mathbb{R}) = Z^r(M, \mathbb{R}) / B^r(M, \mathbb{R})$$

is called the r-th **de Rham cohomology group** of M.

Any smooth map $f : M \to N$ induces a homomorphism

$$f^* : A^r(N) \to A^r(M)$$

which commutes with the coboundary operator d. Such a map f^* is called a **chain map**. It can be easily proved that f^* provides a homomorphism from $Z^r(N, \mathbb{R})$ to $Z^r(M, \mathbb{R})$ and that from $B^r(N, \mathbb{R})$ to $B^r(M, \mathbb{R})$. Hence f^* induces a homomorphism between the de Rham groups

$$f^* : H^r(N, \mathbb{R}) \to H^r(M, \mathbb{R}).$$

3 Connections

3.1 Connections on Vector Bundles

Definition 3.1.1 A **connection** on a vector bundle E is a map

$$\mathrm{D} : \Gamma(E) \to \Gamma(T^*(M) \otimes E)$$

which satisfies the following conditions:

1. For any $s_1, s_2 \in \Gamma(E)$, $\mathrm{D}(s_1 + s_2) = \mathrm{D}s_1 + \mathrm{D}s_2$.

2. For $s \in \Gamma(E)$ and any $\alpha \in C^\infty(M)$, $\mathrm{D}(\alpha s) = \mathrm{d}\alpha \otimes s + \alpha \mathrm{D}s$.

Suppose X is a smooth tangent vector field on M and $s \in \Gamma(E)$. Let

$$\mathrm{D}_X s = \langle X, \mathrm{D}s \rangle,$$

then $\mathrm{D}_X s$ is a section on E, called the **absolute differential quotient** or the **covariant derivative** of the section s along X.

Condition 2 for connections implies that $\mathrm{D}(\lambda s) = \lambda \mathrm{D}s$ for any $\lambda \in \mathbb{R}$, hence D is a linear map from $\Gamma(E)$ to $\Gamma(T^*(M) \otimes E)$. The operator D also has the local property that if the restriction of a section s to an open set $U \subset M$ is zero, then $\mathrm{D}s|_U = 0$. By the definition of absolute differential quotient, it can be shown that for any smooth tangent vector fields X, Y on M, sections s, s_1, s_2 of E, and $\alpha \in C^\infty(M)$, we have

1. $\mathrm{D}_{X+Y} s = \mathrm{D}_X s + \mathrm{D}_Y s$;

2. $\mathrm{D}_{\alpha X} s = \alpha \mathrm{D}_X s$;

3. $\mathrm{D}_X(s_1 + s_2) = \mathrm{D}_X s_1 + \mathrm{D}_X s_2$;

4. $\mathrm{D}_X(\alpha s) = (X\alpha)s + \alpha \mathrm{D}_X s$.

Suppose U is a coordinate neighborhood of M with local coordinates $u^i, 1 \leq i \leq m$. Choose q smooth sections $s_\alpha, 1 \leq \alpha \leq q$ of E on U such that they are linearly independent everywhere. Such a set of sections is called a **local frame field** of E on U. At every point $p \in U$,

$$\{\mathrm{d}u^i \otimes s_\alpha, 1 \leq i \leq m, 1 \leq \alpha \leq q\}$$

forms a basis for the tensor space $T_p^* \otimes E_p$. Since $\mathrm{D}s_\alpha$ is a local section on U of the bundle $T^*(M) \otimes E$, we can write

$$\mathrm{D}s_\alpha = \Gamma_{\alpha i}^\beta \mathrm{d}u^i \otimes s_\beta,$$

where $\Gamma_{\alpha i}^\beta$ are smooth functions on U and the Einstein summation convention is adopted for the indices i and β. Denote

$$\omega_\alpha^\beta = \Gamma_{\alpha i}^\beta \mathrm{d}u^i,$$

then we have

$$Ds_\alpha = \omega_\alpha^\beta \otimes s_\beta.$$

Let $S = (s_1, \cdots, s_q)^T$ and $\omega = (\omega_\alpha^\beta)$, then the above equation can be written as

$$\mathrm{D}S = \omega \otimes S.$$

The matrix ω is called the **connection matrix**, which depends on the choice of the local frame field S.

If $S' = (s_1', \cdots, s_q')^T$ is another local frame field on U, then we may assume that

$$S' = A \cdot S,$$

or equivalently,

$$s_i' = a_i^j s_j,$$

where $A = (a_i^j)$ is a nondegenerate matrix of smooth functions. Suppose the matrix of the connection D with respect to the local frame field S' is ω'. Then we have

$$\begin{aligned}
\mathrm{D}S' &= \mathrm{D}(A \cdot S) \\
&= \mathrm{d}A \otimes S + A \cdot \mathrm{D}S \\
&= (\mathrm{d}A + A \cdot \omega) \otimes S \\
&= (\mathrm{d}A \cdot A^{-1} + A \cdot \omega \cdot A^{-1}) \otimes S'.
\end{aligned}$$

It follows that

$$\omega' = \mathrm{d}A \cdot A^{-1} + A \cdot \omega \cdot A^{-1},$$

or equivalently,

$$\omega' \cdot A = \mathrm{d}A + A \cdot \omega.$$

Conversely, suppose a coordinate covering $\{U_i\}$ is chosen for M. On each U_i fix a local frame field S_i of E and assign a $q \times q$ matrix ω_i of differential 1-forms which satisfies the transformation formula above when the corresponding coordinate neighborhoods intersect. Then there exists a connection D on E whose matrix representation on each member U_i of the coordinate covering is exactly ω_i.

Theorem 3.1.1 A connection always exists on a vector bundle.

Proof. Choose a coordinate covering $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of M. We may assume that there is a local frame field S_α for any U_α. We need only construct a $q \times q$ matrix ω_α on each U_α such that the matrices constructed satisfy the transformation formula under a change of local frame field.

By Theorem 2.3.1 and the Partition of Unity Theorem, we may assume that $\{U_\alpha\}$ is locally finite and $\{g_\alpha\}$ is a corresponding subordinate partition of unity such that $\operatorname{supp} g_\alpha \subset U_\alpha$. When $U_\alpha \cap U_\beta \neq \varnothing$, there naturally exists a nondegenerate matrix $A_{\alpha\beta}$ of smooth functions on $U_\alpha \cap U_\beta$ such that

$$S_\alpha = A_{\alpha\beta} \cdot S_\beta.$$

For every $\alpha \in \mathcal{A}$ choose an arbitrary $q \times q$ matrix φ_α of differential 1-forms on U_α. Let

$$\omega_\alpha = \sum_{U_\alpha \cap U_\beta \neq \varnothing} g_\beta \cdot (\mathrm{d}A_{\alpha\beta} \cdot A_{\alpha\beta}^{-1} + A_{\alpha\beta} \cdot \varphi_\beta \cdot A_{\alpha\beta}^{-1})$$

be another matrix of differential 1-forms on U_α. When $U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing$, we have

$$A_{\alpha\beta} \cdot A_{\beta\gamma} = A_{\alpha\gamma}$$

in the intersection. Thus on $U_\alpha \cap U_\beta \neq \varnothing$, we have

$$\begin{aligned} A_{\alpha\beta} \cdot \omega_\beta \cdot A_{\alpha\beta}^{-1} &= \sum_{U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing} g_\gamma \cdot A_{\alpha\beta} \cdot (\mathrm{d}A_{\beta\gamma} \cdot A_{\beta\gamma}^{-1} + A_{\beta\gamma} \cdot \varphi_\gamma \cdot A_{\beta\gamma}^{-1}) \cdot A_{\alpha\beta}^{-1} \\ &= \sum_{U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing} g_\gamma \cdot (\mathrm{d}A_{\alpha\gamma} - \mathrm{d}A_{\alpha\beta} \cdot A_{\beta\gamma} + A_{\alpha\gamma} \cdot \varphi_\gamma) \cdot A_{\beta\gamma}^{-1} \cdot A_{\alpha\beta}^{-1} \\ &= \omega_\alpha - \mathrm{d}A_{\alpha\beta} \cdot A_{\alpha\beta}^{-1}. \end{aligned}$$

This is precisely the transformation formula. □

Theorem 3.1.2 Suppose D is a connection on a vector bundle E and $p \in M$. Then there exists a local frame field S in a coordinate neighborhood of p such that the corresponding connection matrix ω is zero at p.

Proof. Choose a coordinate neighborhood $(U; u^i)$ of p such that $u^i(p) = 0, 1 \leq i \leq m$. Suppose S' is a local frame field on U with corresponding connection matrix $\omega' = (\omega'^{\beta}_{\alpha})$, where $\omega'^{\beta}_{\alpha} = \Gamma'^{\beta}_{\alpha i} \mathrm{d}u^i$, and the $\Gamma^{\beta}_{\alpha i}$ are smooth functions on U. Let

$$a^{\beta}_{\alpha} = \delta^{\beta}_{\alpha} - \Gamma'^{\beta}_{\alpha i}(p) \cdot u^i.$$

Then $A = (a^{\beta}_{\alpha})$ is the identity matrix at p. Hence there exists a neighborhood $V \subset U$ of p such that A is nondegenerate in V. Thus

$$S = A \cdot S'$$

is a local frame field on V. Noting that

$$\mathrm{d}a^{\beta}_{\alpha} = -\Gamma'^{\beta}_{\alpha i}(p) \cdot \mathrm{d}u^i,$$

we have

$$\mathrm{d}A(p) = -\omega'(p),$$

and hence

$$\omega(p) = \mathrm{d}A(p) \cdot A^{-1}(p) + A(p) \cdot \omega'(p) \cdot A^{-1}(p) = \mathrm{d}A(p) + \omega'(p) = 0.$$

□

Exteriorly differentiating the formula

$$\omega' \cdot A = \mathrm{d}A + A \cdot \omega$$

once, we obtain

$$\mathrm{d}\omega' \cdot A - \omega' \wedge \mathrm{d}A = A \cdot \mathrm{d}\omega + \mathrm{d}A \wedge \omega.$$

Using the formula

$$\mathrm{d}A = \omega' \cdot A - A \cdot \omega,$$

we then have

$$(\mathrm{d}\omega' - \omega' \wedge \omega') \cdot A = A \cdot (\mathrm{d}\omega - \omega \wedge \omega).$$

If we let

$$\Omega = \mathrm{d}\omega - \omega \wedge \omega,$$

then the above equation can be written as

$$\Omega' = A \cdot \Omega \cdot A^{-1}.$$

Definition 3.1.2 The matrix $\Omega = \mathrm{d}\omega - \omega \wedge \omega$ of differential 2-forms is called the **curvature matrix** of the connection D on U.

Choose any two tangent vectors $X_p, Y_p \in T_p(M), p \in U$. Suppose $s_p \in E_p$. Using the local frame field $S_U = (s_1, \cdots, s_q)^T$ of the vector bundle E on U, s_p can be expressed as

$$s_p = \lambda^\alpha s_\alpha|_p.$$

Then let

$$R(X_p, Y_p)s_p = \lambda^\alpha \langle X_p \wedge Y_p, \Omega_\alpha^\beta|_p\rangle s_\beta|_p.$$

Noting that $\langle X_p \wedge Y_p, \Omega_\alpha^\beta|_p\rangle$ is actually a (1, 1)-type tensor on the linear space E_p, $R(X_p, Y_p)$ is a linear transformation on E_p that is independent of the choice of local coordinates.

If X, Y are two smooth tangent vector fields on M, then $R(X, Y)$ is a linear operator on $\Gamma(E)$ given by

$$(R(X, Y)s)_p = R(X_p, Y_p)s_p$$

for each $s \in \Gamma(E), p \in M$. $R(X, Y)$ has the following properties:

1. $R(X, Y) = -R(Y, X)$;
2. $R(fX, Y) = f \cdot R(X, Y)$;
3. $R(X, Y)(fs) = f \cdot R(X, Y)s$,

where $X, Y \in \Gamma(T(M)), f \in C^\infty(M), s \in \Gamma(E)$. $R(X, Y)$ is called the **curvature operator** of the connection D.

Lemma 3.1.3 Suppose ω is a differential 1-form on a smooth manifold M and X, Y are smooth tangent vector fields on M. Then

$$\langle X \wedge Y, \mathrm{d}\omega\rangle = X\langle Y, \omega\rangle - Y\langle X, \omega\rangle - \langle [X, Y], \omega\rangle.$$

Proof. Since both sides are linear with respect to ω, we may assume that ω is a monomial

$$\omega = g\mathrm{d}f,$$

where f, g are smooth functions on M. Therefore

$$\mathrm{d}\omega = \mathrm{d}g \wedge \mathrm{d}f.$$

The left hand side then becomes

$$\langle X\wedge Y, \mathrm{d}\omega\rangle = \langle X\wedge Y, \mathrm{d}g\wedge \mathrm{d}f\rangle = \begin{vmatrix} \langle X, \mathrm{d}g\rangle & \langle X, \mathrm{d}f\rangle \\ \langle Y, \mathrm{d}g\rangle & \langle Y, \mathrm{d}f\rangle \end{vmatrix} = Xg\cdot Yf - Xf\cdot Yg.$$

Since

$$\langle X, \omega\rangle = \langle X, g\mathrm{d}f\rangle = g\cdot Xf,$$

we have

$$Y\langle X, \omega\rangle = Yg\cdot Xf + g\cdot Y(Xf).$$

Similarly,

$$X\langle Y, \omega\rangle = Xg\cdot Yf + g\cdot X(Yf).$$

Therefore the right hand side is also

$$\begin{aligned} & X\langle Y,\omega\rangle - Y\langle X,\omega\rangle - \langle [X,Y],\omega\rangle \\ =& Xg\cdot Yf - Yg\cdot Xf + g\cdot (X(Yf) - Y(Xf)) - g\cdot\langle [X,Y], \mathrm{d}f\rangle \\ =& Xg\cdot Yf - Xf\cdot Yg. \end{aligned}$$

□

Theorem 3.1.4 Suppose X, Y are two arbitrary smooth tangent vector fields on the smooth manifold M. Then

$$R(X,Y) = \mathrm{D}_X\mathrm{D}_Y - \mathrm{D}_Y\mathrm{D}_X - \mathrm{D}_{[X,Y]}.$$

Proof. We need only consider the operators of both sides on a local section. Suppose $s\in\Gamma(E)$ has the local expression

$$s = \lambda^\alpha s_\alpha.$$

Then

$$\mathrm{D}_X s = (X\lambda^\alpha)s_\alpha + \lambda^\alpha \mathrm{D}_X s_\alpha = \left(X\lambda^\alpha + \lambda^\beta\left\langle X, \omega_\beta^\alpha\right\rangle\right)s_\alpha.$$

Hence

$$\begin{aligned} \mathrm{D}_Y\mathrm{D}_X s &= \left[Y(X\lambda^\alpha + \lambda^\beta\left\langle X,\omega_\beta^\alpha\right\rangle) + (X\lambda^\beta + \lambda^\gamma\left\langle X,\omega_\gamma^\beta\right\rangle)\cdot\left\langle Y,\omega_\beta^\alpha\right\rangle\right]s_\alpha \\ &= \left[Y(X\lambda^\alpha) + Y\lambda^\beta\cdot\left\langle X,\omega_\beta^\alpha\right\rangle + \lambda^\beta\cdot Y\left\langle X,\omega_\beta^\alpha\right\rangle\right. \\ &\qquad \left. + X\lambda^\beta\cdot\left\langle Y,\omega_\beta^\alpha\right\rangle + \lambda^\beta\left\langle X,\omega_\beta^\gamma\right\rangle\left\langle Y,\omega_\gamma^\alpha\right\rangle\right]s_\alpha. \end{aligned}$$

It follows that

$$\begin{aligned} (\mathrm{D}_X\mathrm{D}_Y - \mathrm{D}_Y\mathrm{D}_X)s = &\left[[X,Y]\lambda^\alpha + \lambda^\beta\left(X\left\langle Y,\omega_\beta^\alpha\right\rangle - Y\left\langle X,\omega_\beta^\alpha\right\rangle\right.\right. \\ &\left.\left. + \left\langle Y,\omega_\beta^\gamma\right\rangle\left\langle X,\omega_\gamma^\alpha\right\rangle - \left\langle X,\omega_\beta^\gamma\right\rangle\left\langle Y,\omega_\gamma^\alpha\right\rangle\right)\right]s_\alpha. \end{aligned}$$

By Lemma 3.1.3, we have

$$X\left\langle Y,\omega_\beta^\alpha\right\rangle - Y\left\langle X,\omega_\beta^\alpha\right\rangle = \left\langle X\wedge Y, \mathrm{d}\omega_\beta^\alpha\right\rangle + \left\langle [X,Y],\omega_\beta^\alpha\right\rangle.$$

Together with

$$\left\langle X\wedge Y,\omega_\beta^\gamma\wedge\omega_\gamma^\alpha\right\rangle = \left\langle Y,\omega_\beta^\gamma\right\rangle\left\langle X,\omega_\gamma^\alpha\right\rangle - \left\langle X,\omega_\beta^\gamma\right\rangle\left\langle Y,\omega_\gamma^\alpha\right\rangle,$$

we further obtain

$$\begin{aligned}(\mathrm{D}_X\mathrm{D}_Y - \mathrm{D}_Y\mathrm{D}_X)s &= \Big[[X,Y]\lambda^\alpha + \lambda^\beta\left(\left\langle [X,Y],\omega_\beta^\alpha\right\rangle\right.\\ &\qquad\left.+\left\langle X\wedge Y,\mathrm{d}\omega_\beta^\alpha - \omega_\beta^g\wedge\omega_\gamma^\alpha\right\rangle\right)\Big]s_\alpha\\ &= \mathrm{D}_{[X,Y]}s + \lambda^\beta\left\langle X\wedge Y,\Omega_\beta^\alpha\right\rangle s_\alpha\\ &= \left(\mathrm{D}_{[X,Y]} + R(X,Y)\right)s.\end{aligned}$$

That is

$$R(X,Y) = \mathrm{D}_X\mathrm{D}_Y - \mathrm{D}_Y\mathrm{D}_X - \mathrm{D}_{[X,Y]}.$$

□

Theorem 3.1.5 The curvature matrix Ω satisfies the **Bianchi identity**

$$\mathrm{d}\Omega = \omega\wedge\Omega - \Omega\wedge\omega.$$

Proof. Applying exterior differentiation to both sides of

$$\Omega = \mathrm{d}\omega - \omega\wedge\omega,$$

we obtain

$$\begin{aligned}\mathrm{d}\Omega &= -\mathrm{d}\omega\wedge\omega + \omega\wedge\mathrm{d}\omega\\ &= -(\Omega+\omega\wedge\omega)\wedge\omega + \omega\wedge(\Omega+\omega\wedge\omega)\\ &= \omega\wedge\Omega - \Omega\wedge\omega.\end{aligned}$$

□

Definition 3.1.3 Suppose C is a parametrized curve in M, and X is a tangent vector field along C. If a section s of the vector bundle E on C satisfies $\mathrm{D}_X s = 0$, then we say s is **parallel** along the curve C.

Suppose the curve C is given in a local coordinate neighborhood U of M by

$$u^i = u^i(t), \quad 1 \le i \le m.$$

Then the tangent vector field of C is

$$X = \frac{\mathrm{d}u^i}{\mathrm{d}t}\frac{\partial}{\partial u^i}.$$

Let S be a local frame field on U. Then

$$s = \lambda^\alpha s_\alpha$$

is a parallel section along C if and only if it satisfies the system of equations

$$D_X s = \left(\frac{\mathrm{d}\lambda^\alpha}{\mathrm{d}t} + \Gamma^\alpha_{\beta i}\frac{\mathrm{d}u^i}{\mathrm{d}t}\lambda^\beta\right) s_\alpha = 0,$$

or equivalently,

$$\frac{\mathrm{d}\lambda^\alpha}{\mathrm{d}t} + \Gamma^\alpha_{\beta i}\frac{\mathrm{d}u^i}{\mathrm{d}t}\lambda^\beta = 0, \quad 1 \le \alpha \le q.$$

By the Fundamental Theorem of Ordinary Differential Equations, there exists a unique solution for any given initial values. Thus if any vector $v \in E_p$ is given at a point p on C, then it determines uniquely a vector field parallel along C, which is called the **parallel displacement** of v along C.

A connection D of the vector bundle E induces a connection on the dual bundle E^* given by the equation

$$\mathrm{d}\langle s, s^*\rangle = \langle \mathrm{D}s, s^*\rangle + \langle s, \mathrm{D}s^*\rangle$$

for any $s \in \Gamma(E), s^* \in \Gamma(E^*)$. Suppose connections D are separately given on the vector bundles E_1 and E_2, then the equations

$$\mathrm{D}(s_1 \oplus s_2) = \mathrm{D}s_1 \oplus \mathrm{D}s_2$$
$$\mathrm{D}(s_1 \otimes s_2) = \mathrm{D}s_1 \otimes \mathrm{D}s_2$$

determine connections on $E_1 \oplus E_2$ and $E_1 \otimes E_2$, respectively. These are called the **induced connections** on E^*, $E_1 \oplus E_2$ and $E_1 \otimes E_2$, respectively.

3.2 Affine Connections

A connection on the tangent bundle $T(M)$ is called an **affine connection** on the m-dimensional smooth manifold M. A manifold with a given affine connection is called an **affine connection space**.

Suppose M is an m-dimensional affine connection space with a given affine connection D. Choose any coordinate system $(U; u^i)$ of M. Then the natural basis $\{s_i = \partial/\partial u^i, 1 \le i \le m\}$ forms a local frame field of the tangent bundle $T(M)$ on U. Thus we may assume that

$$\mathrm{D}s_i = \omega_i^j \otimes s_j = \Gamma_{ik}^j \mathrm{d}u^k \otimes s_j,$$

where Γ_{ik}^j are smooth functions on U, called the **coefficients** of the connection D with respect to the local coordinates u^i. Suppose $(W; w^i)$ is another coordinate system of M. Let $s_i' = \partial/\partial w^i, 1 \le i \le m$. Then

$$S' = J_{WU} \cdot S$$

holds on $U \cap W \neq \varnothing$, where $J_{WU} = (\partial u^j/\partial w^i), S' = (s_i')^T, S = (s_j)^T$. Then we have

$$\omega' = \mathrm{d}J_{WU} \cdot J_{WU}^{-1} + J_{WU} \cdot \omega \cdot J_{WU}^{-1},$$

or equivalently,

$$\omega_i'^j = \mathrm{d}\left(\frac{\partial u^p}{\partial w^i}\right)\frac{\partial w^j}{\partial u^p} + \frac{\partial u^p}{\partial w^i}\frac{\partial w^j}{\partial u^q}\omega_p^q.$$

Using the relations

$$\omega_i'^j = \Gamma_{ik}'^j \mathrm{d}w^k, \quad \omega_p^q = \Gamma_{pr}^q \mathrm{d}u^r,$$

we obtain

$$\Gamma_{ik}'^j = \Gamma_{pr}^q \frac{\partial w^j}{\partial u^q}\frac{\partial u^p}{\partial w^i}\frac{\partial u^r}{\partial w^k} + \frac{\partial^2 u^p}{\partial w^i \partial w^k}\frac{\partial w^j}{\partial u^p}.$$

This indicates that Γ_{ik}^j is not a tensor field on M.

Suppose X is a smooth vector field on M expressed in local coordinates as

$$X = x^i \frac{\partial}{\partial u^i}.$$

Then

$$\mathrm{D}X = (\mathrm{d}x^i + x^j \omega_j^i) \otimes \frac{\partial}{\partial u^i} = x^i_{,j} \mathrm{d}u^j \otimes \frac{\partial}{\partial u^i},$$

where

$$x^i_{,j} = \frac{\partial x^i}{\partial u^j} + x^k \Gamma_{kj}^i.$$

DX is a tensor field of type (1, 1) on M, called the **absolute differential** of X.

An affine connection on M induces connections on the cotangent bundle $T^*(M)$ and the tensor bundle T^r_s, respectively. Under coordinates u^i, the local coframe field of the cotangent bundle $s^{*i} = \mathrm{d}u^i, 1 \le i \le m$. By the definition of the induced connection on the dual bundle, we have

$$\langle s_j, \mathrm{D}s^{*i}\rangle = \mathrm{d}\langle s_j, s^{*i}\rangle - \langle \mathrm{D}s_j, s^{*i}\rangle = \mathrm{d}\delta^i_j - \omega^i_j = -\omega^i_j$$

for each i, j, hence

$$\mathrm{D}s^{*i} = -\omega^i_j \otimes s^{*j} = -\Gamma^i_{jk}\mathrm{d}u^k \otimes \mathrm{d}u^j.$$

If a cotangent vector field α on M is expressed in local coordinates as

$$\alpha = \alpha_i \mathrm{d}u^i,$$

then

$$\mathrm{D}\alpha = (\mathrm{d}\alpha_i - \alpha_j\omega^j_i) \otimes \mathrm{d}u^i = \alpha_{i,j}\mathrm{d}u^j \otimes \mathrm{d}u^i,$$

where

$$\alpha_{i,j} = \frac{\partial \alpha_i}{\partial u^j} - \alpha_k\Gamma^k_{ij}.$$

Dα is then a (0, 2)-type tensor field, called the **absolute differential** of the cotangent vector field α. In general, if t is an (r, s)-type tensor field, the the image of t under the induced connection D is an $(r, s + 1)$-type tensor field Dt, called the absolute differential of t.

Definition 3.2.1 Suppose $C : u^i = u^i(t)$ is a parametrized curve on M, and $X(t)$ is a tangent vector field defined on C given by

$$X(t) = x^i(t)\left(\frac{\partial}{\partial u^i}\right)_{C(t)}.$$

We say that $X(t)$ is **parallel** along C if its absolute differential along C is zero, i.e. if

$$\frac{\mathrm{D}X}{\mathrm{d}t} = 0.$$

If the tangent vectors of a curve C are parallel along C, then we call C a **self-parallel curve**, or a **geodesic**.

The equation $\mathrm{D}X/\mathrm{d}t = 0$ is equivalent to

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} + x^j\Gamma^i_{jk}\frac{\mathrm{d}u^k}{\mathrm{d}t} = 0.$$

This is a system of first-order ordinary differential equations. Thus a given tangent vector X at any point on C gives rise to a parallel tangent vector field, called the **parallel displacement** of X along the curve C.

If C is a geodesic, then its tangent vector

$$X(t) = \frac{\mathrm{d}u^i(t)}{\mathrm{d}t}\left(\frac{\partial}{\partial u^i}\right)_{C(t)}$$

is parallel along C. Therefore a geodesic curve C should satisfy

$$\frac{\mathrm{d}^2u^i}{\mathrm{d}t^2} + \Gamma^i_{jk}\frac{\mathrm{d}u^j}{\mathrm{d}t}\frac{\mathrm{d}u^k}{\mathrm{d}t} = 0.$$

This is a system of second-order ordinary differential equations. Thus there exists a unique geodesic through a given point of M which is tangent to a given tangent vector at that point.

Now consider the curvature matrix Ω of an affine connection. Since

$$\omega^j_i = \Gamma^j_{ik}\mathrm{d}u^k,$$

we have

$$\begin{aligned}\mathrm{d}\omega^j_i - \omega^h_i \wedge \omega^j_h &= \frac{\partial\Gamma^j_{ik}}{\partial u^l}\mathrm{d}u^l \wedge \mathrm{d}u^k - \Gamma^h_{il}\Gamma^j_{hk}\mathrm{d}u^l \wedge \mathrm{d}u^k \\ &= \frac{1}{2}\left(\frac{\partial\Gamma^j_{il}}{\partial u^k} - \frac{\partial\Gamma^j_{ik}}{\partial u^l} + \Gamma^h_{il}\Gamma^j_{hk} - \Gamma^h_{ik}\Gamma^j_{hl}\right)\mathrm{d}u^k \wedge \mathrm{d}u^l.\end{aligned}$$

Therefore

$$\Omega^j_i = \frac{1}{2}R^j_{ikl}\mathrm{d}u^k \wedge \mathrm{d}u^l,$$

where

$$R^j_{ikl} = \frac{\partial\Gamma^j_{il}}{\partial u^k} - \frac{\partial\Gamma^j_{ik}}{\partial u^l} + \Gamma^h_{il}\Gamma^j_{hk} - \Gamma^h_{ik}\Gamma^j_{hl}.$$

If $(W; w^i)$ is another coordinate system of M, then

$$\Omega' = J_{WU} \cdot \Omega \cdot J^{-1}_{WU},$$

where Ω' is the curvature matrix of the connection D under the coordinate system $(W; w^i)$. Therefore

$$\Omega_i'^j = \Omega_p^q \frac{\partial u^p}{\partial w^i} \frac{\partial w^j}{\partial u^q}.$$

Thus

$$R_{ikl}'^j = R_{prs}^q \frac{\partial u^p}{\partial w^i} \frac{\partial w^j}{\partial u^q} \frac{\partial u^r}{\partial w^k} \frac{\partial u^s}{\partial w^l},$$

where $R_{ikl}'^j$ is determined by

$$\Omega_i'^j = \frac{1}{2} R_{ikl}'^j \mathrm{d}w^k \wedge \mathrm{d}w^l.$$

If we let

$$R = R_{ikl}^j \mathrm{d}u^i \otimes \frac{\partial}{\partial u^j} \otimes \mathrm{d}u^k \otimes \mathrm{d}u^l,$$

then R is independent of the choice of local coordinates, and is called the **curvature tensor** of the affine connection.

Suppose X, Y, Z are tangent vector field with local expressions

$$X = X^i \frac{\partial}{\partial u^i}, \quad Y = Y^i \frac{\partial}{\partial u^i}, \quad Z = Z^i \frac{\partial}{\partial u^i}.$$

Then by the definition of the curvature operator, we have

$$R(X,Y)Z = Z^i \left\langle X \wedge Y, \Omega_i^j \right\rangle \frac{\partial}{\partial u^j} = R_{ikl}^j Z^i X^k Y^l \frac{\partial}{\partial u^j}.$$

Thus

$$R_{ikl}^j = \left\langle R\left(\frac{\partial}{\partial u^k}, \frac{\partial}{\partial u^l}\right) \frac{\partial}{\partial u^i}, \mathrm{d}u^j \right\rangle.$$

This is the relation between the curvature operator and the curvature tensor.

The connection coefficients Γ_{ik}^j does not satisfy the transformation rule for tensors. But if we define $T_{ik}^j = \Gamma_{ki}^k - \Gamma_{ik}^j$, then we have

$$T_{ik}'^j = T_{pr}^q \frac{\partial w^j}{\partial u^q} \frac{\partial u^p}{\partial w^i} \frac{\partial u^r}{\partial w^k}$$

after the transformation formula for Γ_{ik}^j. Thus

$$T = T_{ik}^j \frac{\partial}{\partial u^j} \otimes \mathrm{d}u^i \otimes \mathrm{d}u^k$$

is a (1, 2)-type tensor, called the **torsion tensor** of the affine connection D. T can also be viewed as a map from $\Gamma(T(M)) \times \Gamma(T(M))$ to $\Gamma(T(M))$. Suppose X, Y are any two tangent vector field on M. Then $T(X, Y)$ is a tangent vector field on M with local expression

$$T(X,Y) = T^k_{ij} X^i Y^j \frac{\partial}{\partial u^k}.$$

It can be verified that

$$T(X,Y) = \mathrm{D}_X Y - \mathrm{D}_Y X - [X,Y].$$

Definition 3.2.2 If the torsion tensor of an affine connection D is zero, then the connection is said to be **torsion-free**.

If the coefficients of a connection D are Γ^j_{ik}, then set

$$\tilde{\Gamma}^j_{ik} = \frac{1}{2}(\Gamma^j_{ik} + \Gamma^j_{ki}).$$

The $\tilde{\Gamma}^j_{ik}$ can be the coefficients of some connection $\tilde{\mathrm{D}}$ since they satisfy the transformation formula for connection coefficients, and direct computation suggests that $\tilde{\mathrm{D}}$ is torsion-free. Therefore a torsion-free connection on a vector bundle always exists. Noting that

$$\Gamma^j_{ik} = -\frac{1}{2} T^j_{ik} + \tilde{\Gamma}^j_{ik},$$

we have

$$\mathrm{D}_X Z = \frac{1}{2} T(X,Z) + \tilde{\mathrm{D}}_X Z.$$

This implies that any connection can be decomposed into a sum of a multiple of its torsion tensor and a torsion-free connection. Moreover, since the geodesic equation of the connection D is equivalent to

$$\frac{\mathrm{d}^2 u^i}{\mathrm{d}t^2} + \tilde{\Gamma}^i_{jk} \frac{\mathrm{d}u^j}{\mathrm{d}t} \frac{\mathrm{d}u^k}{\mathrm{d}t} = 0,$$

a connection D and the corresponding torsion-free connection $\tilde{\mathrm{D}}$ have the same geodesics.

Theorem 3.2.1 Suppose D is a torsion-free affine connection on M. Then for any point $p \in M$ there exists a local coordinate system u^i such that the corresponding connection coefficients Γ^j_{ik} vanish at p.

Proof. Suppose $(W; w^i)$ is a local coordinate system at p with connection coefficients Γ'^j_{ik}. Let

$$u^i = w^i + \frac{1}{2}\Gamma'^i_{jk}(p)(w^j - w^j(p))(w^k - w^k(p)).$$

Then

$$\left.\frac{\partial u^i}{\partial w^j}\right|_p = \delta^i_j, \qquad \left.\frac{\partial^2 u^i}{\partial w^j \partial w^k}\right|_p = \Gamma'^i_{jk}(p).$$

Thus the matrix $(\partial u^i/\partial w^j)$ is nondegenerate at p, and then the u^i provide a local coordinates in a neighborhood of p. Then the connection coefficients Γ^j_{ik} in the new coordinate system u^i satisfy

$$\Gamma^j_{ik}(p) = 0, \quad 1 \le i, j, k \le m.$$

□

Theorem 3.2.2 Suppose D is a torsion-free affine connection on M. Then we have the Bianchi identity

$$R^j_{ikl,h} + R^j_{ilh,k} + R^j_{ihk,l} = 0,$$

where $R^j_{ikl,h}$ is determined by the absolute differential of the curvature tensor R as

$$\mathrm{D}R = R^j_{ikl,h} \mathrm{d}u^h \otimes \mathrm{d}u^i \otimes \frac{\partial}{\partial u^j} \otimes \mathrm{d}u^k \otimes \mathrm{d}u^l.$$

Proof. From Theorem 3.1.5 we have

$$\mathrm{d}\Omega^j_i = \omega^k_i \wedge \Omega^j_k - \Omega^k_i \wedge \omega^j_k,$$

that is

$$\frac{\partial R^j_{ikl}}{\partial u^h} \mathrm{d}u^h \wedge \mathrm{d}u^k \wedge \mathrm{d}u^l = (\Gamma^p_{ih} R^j_{pkl} - \Gamma^j_{ph} R^p_{ikl}) \mathrm{d}u^h \wedge \mathrm{d}u^k \wedge \mathrm{d}u^l.$$

From

$$R = R^j_{ikl} \mathrm{d}u^i \otimes \frac{\partial}{\partial u^j} \otimes \mathrm{d}u^k \otimes \mathrm{d}u^l,$$

we obtain

$$
\begin{aligned}
& \mathrm{D}R \\
=\; & \mathrm{d}R^j_{ikl} \otimes \mathrm{d}u^i \otimes \frac{\partial}{\partial u^j} \otimes \mathrm{d}u^k \otimes \mathrm{d}u^l + R^j_{ikl}\mathrm{D}(\mathrm{d}u^i) \otimes \frac{\partial}{\partial u^j} \otimes \mathrm{d}u^k \otimes \mathrm{d}u^l \\
& + R^j_{ikl}\mathrm{d}u^i \otimes \mathrm{D}\left(\frac{\partial}{\partial u^j}\right) \otimes \mathrm{d}u^k \otimes \mathrm{d}u^l + R^j_{ikl}\mathrm{d}u^i \otimes \frac{\partial}{\partial u^j} \otimes \mathrm{D}(\mathrm{d}u^k) \otimes \mathrm{d}u^l \\
& + R^j_{ikl}\mathrm{d}u^i \otimes \frac{\partial}{\partial u^j} \otimes \mathrm{d}u^k \otimes \mathrm{D}(\mathrm{d}u^l) \\
=\; & (\mathrm{d}R^j_{ikl} - R^j_{pkl}\omega^p_i + R^p_{ikl}\omega^j_p - R^j_{ipl}\omega^p_k - R^j_{ikp}\omega^p_l) \otimes \frac{\partial}{\partial u^j} \otimes \mathrm{d}u^i \otimes \mathrm{d}u^k \otimes \mathrm{d}u^l,
\end{aligned}
$$

and hence

$$
R^j_{ikl,h} = \frac{\partial R^j_{ikl}}{\partial u^h} - \Gamma^p_{ih}R^j_{pkl} + \Gamma^j_{ph}R^p_{ikl} - \Gamma^p_{kh}R^j_{ipl} - \Gamma^p_{lh}R^j_{ikp}.
$$

Therefore

$$
\begin{aligned}
& R^j_{ikl,h}\mathrm{d}u^h \wedge \mathrm{d}u^k \wedge \mathrm{d}u^l \\
=\; & \left(\frac{\partial R^j_{ikl}}{\partial u^h} + \Gamma^j_{ph}R^p_{ikl} - \Gamma^p_{ih}R^j_{pkl} - \Gamma^p_{kh}R^j_{ipl} - \Gamma^p_{lh}R^j_{ikp}\right)\mathrm{d}u^h \wedge \mathrm{d}u^k \wedge \mathrm{d}u^l \\
=\; & -(\Gamma^p_{kh}R^j_{ipl} + \Gamma^p_{lh}R^j_{ikp})\mathrm{d}u^h \wedge \mathrm{d}u^k \wedge \mathrm{d}u^l.
\end{aligned}
$$

The torsion-free property of the connection implies that

$$
\Gamma^p_{lh}R^j_{ikp}\mathrm{d}u^h \wedge \mathrm{d}u^k \wedge \mathrm{d}u^l = \Gamma^p_{hk}R^j_{ilp}\mathrm{d}u^h \wedge \mathrm{d}u^k \wedge \mathrm{d}u^l = -\Gamma^p_{kh}R^j_{ipl}\mathrm{d}u^h \wedge \mathrm{d}u^k \wedge \mathrm{d}u^l,
$$

thus

$$
R^j_{ikl,h}\mathrm{d}u^h \wedge \mathrm{d}u^k \wedge \mathrm{d}u^l = 0.
$$

Hence

$$
(R^j_{ikl,h} + R^j_{ilh,k} + R^j_{ihk,l})\mathrm{d}u^h \wedge \mathrm{d}u^k \wedge \mathrm{d}u^l = 0.
$$

Since the coefficients are skew-symmetric with respect to h, k, l, we have

$$
R^j_{ikl,h} + R^j_{ilh,k} + R^j_{ihk,l} = 0.
$$

□

3.3 Connections on Frame Bundles

Suppose M is an m-dimensional differentiable manifold. A **frame** refers to a combination of the form $(p; e_1, \cdots, e_m)$, where p is a point in M and $e_1, \cdots, e_m$ are m linearly independent tangent vectors at p. The set of all frames on M is denoted by P. We now introduce a differentiable structure on P so that it becomes a smooth manifold, and the natural projection

$$\pi(p; e_1, \cdots, e_m) = p$$

is a smooth map from P to M. (P, M, π) is then called the **frame bundle** of M.

Suppose $(U; u^i)$ is a coordinate neighborhood of M. Then there is a natural frame field $(\partial/\partial u^1, \cdots, \partial/\partial u^m)$ on U. Hence any frame $(p; e_1, \cdots, e_m)$ on U can be written as

$$e_i = X_i^k \left(\frac{\partial}{\partial u^k}\right)_p, \quad 1 \le i \le m,$$

where (X_i^k) is a nondegenerate $m \times m$ matrix, and therefore an element of $\mathrm{GL}(m; \mathbb{R})$. Thus we can define a map $\varphi_U : U \times \mathrm{GL}(m; \mathbb{R}) \to \pi^{-1}(U)$ by

$$\varphi_U(p, X_i^k) = \left(p; X_1^k \left(\frac{\partial}{\partial u^k}\right)_p, \cdots, X_m^k \left(\frac{\partial}{\partial u^k}\right)_p\right)$$

for any $p \in U, (X_i^k) \in \mathrm{GL}(m; \mathbb{R})$. We can see that φ_U is a one-to-one correspondence.

Choose a coordinate covering $\{U_1, U_2, \cdots\}$ of M with corresponding maps $\{\varphi_1, \varphi_2, \cdots\}$. The images of all the open subsets of $U_i \times \mathrm{GL}(m; \mathbb{R})$ under the map φ_i form a topological basis for P. With respect to this topological structure of P, the map $\varphi_i : U_i \times \mathrm{GL}(m; \mathbb{R}) \to \pi^{-1}(U)$ is a homeomorphism.

Through the map φ_i, $\pi^{-1}(U_i)$ becomes a coordinate neighborhood in P with local coordinate system (u^i, X_i^k). Suppose U and W are coordinate neighborhoods in M such that $U \cap W \neq \varnothing$. Then M has the local change of coordinates

$$w^i = w^i(u^1, \cdots, u^m), \quad 1 \le i \le m$$

on the intersection $U \cap W$. The corresponding natural bases have the following relationship

$$\frac{\partial}{\partial u^i} = \frac{\partial w^j}{\partial u^i} \frac{\partial}{\partial w^j}.$$

If $(p; e_1, \cdots, e_m)$ is a frame on $U \cap W$, then its coordinates (u^i, X_i^k) and (w^i, Y_i^k) under two coordinate systems satisfy

$$w^i = w^i(u^1, \cdots, u^m), \quad 1 \le i \le m,$$

and

$$Y_i^k = X_i^j \frac{\partial w^k}{\partial u^j}, \quad 1 \le i, k \le m.$$

We can then see that the coordinate neighborhoods $\pi^{-1}(U)$ and $\pi^{-1}(W)$ are C^∞-compatible. Therefore P becomes an (m^2+m)-dimensional smooth manifold, and the natural projection $\pi : P \to M$ is a smooth surjection.

For any $p \in U$, let

$$\varphi_{U,p}(X) = \varphi_U(p, X), \quad X \in \mathrm{GL}(m; \mathbb{R}).$$

Then $\varphi_{U,p} : \mathrm{GL}(m; \mathbb{R}) \to \pi^{-1}(p)$ is a homeomorphism. If $U \cap W \neq \varnothing$, for $p \in U \cap W$, the map $\varphi_{W,p}^{-1} \circ \varphi_{U,p}$ is a homeomorphism from $\mathrm{GL}(m; \mathbb{R})$ to itself. In fact, $\varphi_{W,p}^{-1} \circ \varphi_{U,p}$ is precisely the right translation of the Jacobian matrix $J_{UW} = (\partial w^k / \partial u^j)$ on $\mathrm{GL}(m; \mathbb{R})$. Thus $\{J_{UW}\}$ forms a family of transition functions on the frame bundle. Therefore the frame bundle P is a fiber bundle that is not a vector bundle with $\mathrm{GL}(m; \mathbb{R})$ as its typical fiber.

Suppose $(U; u^i)$ and $(W; w^i)$ are two coordinate systems on M with the corresponding coordinate systems (u^i, X_i^k) and (w^i, Y_i^k) on P. Use (X_i^{*k}) and (Y_i^{*k}) to denote the inverse matrices of (X_i^k) and (Y_i^k), respectively, that is

$$X_i^k X_k^{*j} = X_i^{*k} X_k^j = \delta_i^j, \quad Y_i^k Y_k^{*j} = Y_i^{*k} Y_k^j = \delta_i^j.$$

If $U \cap W \neq \varnothing$, then on $U \cap W$ we have

$$\mathrm{d}w^i = \frac{\partial w^i}{\partial u^j} \mathrm{d}u^j.$$

On the other hand, since

$$Y_i^k = X_i^j \frac{\partial w^k}{\partial u^j},$$

we have

$$X_i^{*j} = \frac{\partial w^k}{\partial u^i} Y_k^{*j}.$$

Hence

$$X_i^{*j} \mathrm{d}u^i = Y_k^{*j} \frac{\partial w^k}{\partial u^i} \mathrm{d}u^i = Y_k^{*j} \mathrm{d}w^k.$$

This implies that the differential 1-form

$$\theta^i = X_j^{*i} \mathrm{d}u^j$$

is independent of the choice of local coordinates of P. Therefore θ^i can be defined to be a differential 1-form on P.

Now suppose M is an m-dimensional affine connection space with connection D. Suppose the connection matrix of D under the local coordinate system $(U; u^i)$ is $\omega = (\omega_i^j)$. Then the absolute differential of the vector field $e_i = X_k^i(\partial/\partial u^k)$ is

$$\mathrm{D}e_i = (\mathrm{d}X_i^k + X_i^j \omega_i^k) \otimes \frac{\partial}{\partial u^k}.$$

If we view X_i^k as independent variables and let

$$\mathrm{D}X_i^k = \mathrm{d}X_i^k + X_i^j \omega_j^k,$$

then $\mathrm{D}X_i^k$ is a differential 1-form on the coordinate neighborhood $\pi^{-1}(U)$ on P. Suppose $(W; w^i)$ is another local coordinate system of M. If $U \cap W \neq \varnothing$, then we have

$$Y_i^k = X_i^j \frac{\partial w^k}{\partial u^j}$$

on $U \cap W$. Thus

$$\begin{aligned}
\mathrm{D}Y_i^k &= \mathrm{d}Y_i^k + Y_i^j \omega_j'^k \\
&= \mathrm{d}X_i^j \cdot \frac{\partial w^k}{\partial u^j} + X_i^j \mathrm{d}\left(\frac{\partial w^k}{\partial u^j}\right) + \\
&\quad X_i^l \frac{\partial w^j}{\partial u^l} \left[\mathrm{d}\left(\frac{\partial u^p}{\partial w^j}\right) \frac{\partial w^k}{\partial u^p} + \frac{\partial u^p}{\partial w^j} \frac{\partial w^k}{\partial u^q} \omega_p^q\right] \\
&= \left(\mathrm{d}X_i^j + X_i^l \omega_l^j\right) \frac{\partial w^k}{\partial u^j} \\
&= \mathrm{D}X_i^j \cdot \frac{\partial w^k}{\partial u^j}.
\end{aligned}$$

Hence

$$Y_k^{*j} \mathrm{D}Y_i^k = Y_k^{*j} \frac{\partial w^k}{\partial u^l} \mathrm{D}X_i^l = X_l^{*j} \mathrm{D}X_i^l.$$

It follows that the differential 1-form

$$\theta_i^j = X_k^{*j} \mathrm{D}X_i^k = X_k^{*j} \left(\mathrm{d}X_i^k + X_i^l \omega_l^k\right)$$

is independent of the choice of the local coordinate system, and is therefore a differential 1-form on P.

Because (u^i, X_i^k) is a local coordinate system on P, $(\mathrm{d}u^i, \mathrm{d}X_i^k)$ are coordinates of the cotangent space at a point in P. Now θ^i along with θ_i^j are (m^2+m) differential 1-forms defined on P. They can be written as linear combinations of $\mathrm{d}u^i, \mathrm{d}X_i^k$ in the coordinate neighborhood $\pi^{-1}(U)$, and vice versa. Thus θ^i and θ_i^k are linearly independent everywhere, that is $\{\theta^i, \theta_i^k\}$ forms a coframe field on the whole of P, whose dual is then a global frame field on P.

Under the local coordinate system $(U; u^i)$, we have

$$\begin{aligned}\mathrm{d}u^i &= X_j^i \theta^j,\\ \mathrm{d}X_i^j &= -X_i^k \omega_k^j + X_k^j \theta_i^k,\end{aligned}$$

after the definition of θ^i and θ_i^k. Exteriorly differentiating both equations, we obtain

$$\begin{aligned}0 &= \mathrm{d}X_j^i \wedge \theta^j + X_j^i \mathrm{d}\theta^j\\ &= \left(-X_j^k \omega_k^i + X_k^i \theta_j^k\right) \wedge \theta^j + X_j^i \mathrm{d}\theta^j\\ &= \left(-X_j^k \Gamma_{kl}^i X_h^l \theta^h + X_k^i \theta_j^k\right) \wedge \theta^j + X_j^i \mathrm{d}\theta^j\\ &= X_j^i \left(\mathrm{d}\theta^j - \theta^k \wedge \theta_k^j\right) - X_k^p X_h^l \Gamma_{pl}^i \theta^h \wedge \theta^k,\end{aligned}$$

and

$$\begin{aligned}0 &= -\mathrm{d}X_i^k \wedge \omega_k^j - X_i^k \mathrm{d}\omega_k^j + \mathrm{d}X_k^j \wedge \theta_i^k + X_k^j \mathrm{d}\theta_i^k\\ &= -\left(-X_i^l \omega_l^k + X_l^k \theta_i^l\right) \wedge \omega_k^j - X_i^k \mathrm{d}\omega_k^j\\ &\quad + \left(-X_k^l \omega_l^j + X_l^j \theta_k^l\right) \wedge \theta_i^k + X_k^j \mathrm{d}\theta_i^k\\ &= -X_i^k \Omega_k^j + X_k^j \left(\mathrm{d}\theta_i^k - \theta_i^l \wedge \theta_l^k\right).\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{d}\theta^j - \theta^k \wedge \theta_k^j &= X_r^{*j} X_k^p X_h^l \Gamma_{pl}^r \theta^h \wedge \theta^k\\ &= \frac{1}{2} X_r^{*j} X_k^p X_l^q T_{pq}^r \theta^k \wedge \theta^l,\end{aligned}$$

and

$$\begin{aligned}\mathrm{d}\theta_i^j - \theta_i^k \wedge \theta_k^j &= X_h^{*j} X_i^k \Omega_k^h\\ &= \frac{1}{2} X_q^{*j} X_i^p X_k^r X_l^s R_{prs}^q \theta^k \wedge \theta^l.\end{aligned}$$

Here T^r_{pq} and R^q_{prs} are, respectively, the torsion tensor and the curvature tensor. Let

$$P^j_{kl} = X^{*j}_r X^p_k X^q_l T^r_{pq},$$
$$S^j_{ikl} = X^{*j}_q X^p_i X^r_k X^s_l R^q_{prs}.$$

Then the above equations become

$$\mathrm{d}\theta^j - \theta^k \wedge \theta^j_k = \frac{1}{2} P^j_{kl} \theta^k \wedge \theta^l,$$
$$\mathrm{d}\theta^j_i - \theta^k_i \wedge \theta^j_k = \frac{1}{2} S^j_{ikl} \theta^k \wedge \theta^l.$$

Obviously P^j_{kl} and S^j_{ikl} are independent of the choice of local coordinates. Therefore the above equations are valid on the whole frame bundle P, and comprise the so-called **structure equations** of the connection.

The differential forms θ^i are determined by the differentiable structure of M. The importance of the structure equations is that collectively they give a sufficient condition for the m^2 differential forms θ^k_i to define an affine connection on M.

Lemma 3.3.1 (Cartan's Lemma) Suppose $\{v_1, \cdots, v_r\}$ and $\{w_1, \cdots, w_r\}$ are two sets of vectors in V such that

$$\sum_{i=1}^{r} v_i \wedge w_i = 0.$$

If $v_1, \cdots, v_r$ are linearly independent, then the w_i can be expressed as linear combinations of the v_j:

$$w_i = \sum_{j=1}^{r} a_{ij} v_j, \quad 1 \le i \le r,$$

with $a_{ij} = a_{ji}$.

Theorem 3.3.2 Suppose $\theta^j_i, 1 \le i, j \le m$ are m^2 differential 1-forms on the frame bundle P. If they and the θ^i satisfy the structure equation

$$\mathrm{d}\theta^j - \theta^k \wedge \theta^j_k = \frac{1}{2} P^j_{kl} \theta^k \wedge \theta^l,$$
$$\mathrm{d}\theta^j_i - \theta^k_i \wedge \theta^j_k = \frac{1}{2} S^j_{ikl} \theta^k \wedge \theta^l,$$

where P^j_{kl} and S^j_{ikl} are certain functions defined in P, then there exists an affine connection D on M such that θ^j_i and D are related as

$$\theta^j_i = X^{*j}_k \mathrm{D} X^k_i$$

locally.

Proof. Choose a coordinate neighborhood $(U; u^i)$ of M, then (u^i, X^k_i) is a local coordinate system in P. Then

$$\theta^i = X^{*i}_k \mathrm{d}u^k,$$

where (X^{*i}_k) is the inverse matrix of (X^k_i). Therefore

$$\mathrm{d}\theta^i = \mathrm{d}X^{*i}_k \wedge \mathrm{d}u^k = \left(\mathrm{d}X^{*i}_k \cdot X^k_j\right) \wedge \theta^j = -X^{*i}_k \mathrm{d}X^k_j \wedge \theta^j.$$

Plugging this into the structure equation we have

$$\theta^j \wedge \left(\theta^i_j + \frac{1}{2} P^i_{jk}\theta^k - X^{*i}_k \mathrm{d}X^k_j\right) = 0.$$

Since the θ^j are linearly independent, by Cartan's Lemma, $\theta^i_j - X^{*i}_k \mathrm{d}X^k_j$ are linear combinations of the θ^l. Thus we may assume

$$X^k_j \theta^j_i - \mathrm{d}X^k_i = \omega^k_j X^j_i,$$

where ω^k_j are linear combinations of θ^l, and hence of $\mathrm{d}u^i$. Let

$$\omega^k_j = \Gamma^k_{ji} \mathrm{d}u^i,$$

where Γ^k_{ji} are functions on P. If we can show that the Γ^k_{ji} are functions of u^i only and independent of X^j_i, then Γ^k_{ji} are the coefficients of some connection under the local coordinates u^i, and the theorem will be proved.

Exteriorly differentiating the equation

$$X^k_j \theta^j_i - \mathrm{d}X^k_i = \omega^k_j X^j_i$$

we obtain

$$\mathrm{d}X^k_j \wedge \theta^j_i + X^k_j \mathrm{d}\theta^j_i = \mathrm{d}\omega^k_j \cdot X^j_i - \omega^k_j \wedge \mathrm{d}X^j_i.$$

This can be simplified to

$$X^j_i \left(\mathrm{d}\omega^k_j - \omega^l_j \wedge \omega^k_l\right) = \frac{1}{2} X^k_j S^j_{ilh} \theta^l \wedge \theta^h$$

by the structure equation. Since the right hand side contains only the differentials $\mathrm{d}u^i$ and so does $\omega_j^l \wedge \omega_l^k$, $\mathrm{d}\omega_j^k$ should also contain only the differentials $\mathrm{d}u^i$. From

$$\omega_j^k = \Gamma_{ji}^k \mathrm{d}u^i$$

we have

$$\mathrm{d}\omega_j^k = \frac{\partial \Gamma_{ji}^k}{\partial u^l} \mathrm{d}u^l \wedge \mathrm{d}u^i + \frac{\partial \Gamma_{ji}^k}{\partial X_l^h} \mathrm{d}X_l^h \wedge \mathrm{d}u^i.$$

Hence

$$\frac{\partial \Gamma_{ji}^k}{\partial X_l^h} = 0.$$

Therefore Γ_{ji}^k are only functions of u^i.

Suppose $(W; w^i)$ is another coordinate neighborhood of M. Then (w^i, Y_i^k) is the local coordinate system of P in $\pi^{-1}(W)$. If $U \cap W \neq \varnothing$, then on $U \cap W$ we have

$$\theta_i^j = X_k^{*j} \left(\mathrm{d}X_i^k + X_i^l \omega_l^k \right) = Y_k^{*j} \left(\mathrm{d}Y_i^k + Y_i^l \omega_l'^k \right),$$

where $\omega_l'^k = \Gamma_{lj}'^k \mathrm{d}w^j$ and the $\Gamma_{lj}'^k$ are functions of w^j only. Plugging

$$Y_i^k = X_i^j \frac{\partial w^k}{\partial u^j}$$

and

$$X_i^{*j} = \frac{\partial w^k}{\partial u^i} Y_k^{*j}$$

into this equation, we get

$$\omega_i'^j = \mathrm{d}\left(\frac{\partial u^p}{\partial w^i} \right) \frac{\partial w^j}{\partial u^p} + \frac{\partial u^p}{\partial w^i} \frac{\partial w^j}{\partial u^q} \omega_p^q.$$

This implies that (ω_i^j) indeed defines an affine connection D on M, such that (ω_i^j) is the connection matrix of D under the local coordinate system $(U; u^i)$. □

4 Riemannian Geometry

4.1 The Fundamental Theorem of Riemannian Geometry

Suppose M is an m-dimensional smooth manifold, and G is a symmetric covariant tensor field of rank 2 on M. If $(U; u^i)$ is a local coordinate system on M, then the tensor field G can be expressed as

$$G = g_{ij} \mathrm{d}u^i \otimes \mathrm{d}u^j$$

on U, where $g_{ij} = g_{ji}$ is a smooth function on U. G provides a bilinear function on $T_p(M)$ at every point $p \in M$. Suppose

$$X = X^i \frac{\partial}{\partial u^i}, \quad Y = Y^i \frac{\partial}{\partial u^i},$$

then

$$G(X, Y) = g_{ij} X^i Y^j.$$

We say that the tensor G is **nondegenerate** at the point p if, whenever $X \in T_p(M)$ and $G(X, Y) = 0$ for all $Y \in T_p(M)$, it must be true that $X = 0$. This implies that G is nondegenerate at p if and only if $\det(g_{ij}(p)) \neq 0$. If for all $X \in T_p(M)$ we have $G(X, X) \geq 0$ and the equality holds only if $X = 0$, then we say G is **positive definite** at p. A positive definite tensor G is necessarily nondegenerate.

Definition 4.1.1 If an m-dimensional smooth manifold M is given a smooth, everywhere nondegenerate symmetric covariant tensor field G of rank 2, then M is called a **generalized Riemmanian manifold**, and G is called a **fundamental tensor** of **metric tensor** of M. If G is positive definite, then M is called a **Riemannian manifold**.

For a generalized Riemannian manifold M, G specifies an inner product on the tangent space $T_p(M)$ at every point $p \in M$. For any $X, Y \in T_p(M)$, let

$$X \cdot Y = G(X, Y) = g_{ij}(p) X^i Y^j.$$

When G is positive definite, it is meaningful to define the length of a tangent vector and the angle between two tangent vectors at the same point, i.e.,

$$|X| = \sqrt{g_{ij} X^i Y^j}, \quad \cos \angle(X, Y) = \frac{X \cdot Y}{|X||Y|}.$$

Thus a Riemannian manifold is a differentiable manifold which has a positive definite inner product on the tangent space at every point. The inner product is required to be smooth in the sense that if X, Y are smooth tangent vector fields, then $X \cdot Y$ is a smooth function on M.

The differential 2-form

$$\mathrm{d}s^2 = g_{ij} \mathrm{d}u^i \mathrm{d}u^j$$

is independent of the choice of the local coordinate system u^i and is usually called the **metric form** or **Riemannian metric**. $\mathrm{d}s$ is precisely the length of an infinitesimal tangent vector, and is called the **element of arc length**.

Suppose $C : u^i = u^i(t), t_0 \le t \le t_1$ is a continuous and piecewise smooth parametrized curve on M. Then the arc length of C is defined to be

$$s = \int_{t_0}^{t_1} \sqrt{g_{ij} \frac{\mathrm{d}u^i}{\mathrm{d}t} \frac{\mathrm{d}u^j}{\mathrm{d}t}} \mathrm{d}t.$$

Theorem 4.1.1 There exists a Riemannian metric on any m-dimensional smooth manifold M.

Proof. Choose a locally finite coordinate covering $\{(U_\alpha; u_\alpha^i)\}$ of M. Suppose $\{h_\alpha\}$ is the corresponding partition of unity such that $\operatorname{supp} h_\alpha \subset U_\alpha$. Let

$$\mathrm{d}s_\alpha^2 = \sum_{i=1}^m (\mathrm{d}u_\alpha^i)^2, \quad \mathrm{d}s^2 = \sum_\alpha h_\alpha \mathrm{d}s_\alpha^2.$$

Then the $\mathrm{d}s_\alpha^2$ and $\mathrm{d}s^2$ are defined to be smooth differential 2-forms on M. If we choose a coordinate neighborhood $(U; u^i)$ such that $\overline{U}$ is compact, then U intersects only finitely many $U_{\alpha_1}, \cdots, U_{\alpha_r}$ by the local finiteness of $\{U_\alpha\}$. Thereforethe restriction of $\mathrm{d}s^2$ to U is

$$\mathrm{d}s^2 = \sum_{\lambda=1}^r h_{\alpha_\lambda} \mathrm{d}s_{\alpha_\lambda}^2 = g_{ij} \mathrm{d}u^i \mathrm{d}u^j,$$

where

$$g_{ij} = \sum_{\lambda=1}^r \sum_{k=1}^m h_{\alpha_\lambda} \frac{\partial u_{\alpha_\lambda}^k}{\partial u^i} \frac{\partial u_{\alpha_\lambda}^k}{\partial u^j}.$$

Since $0 \le h_\alpha \le 1$ and $\sum_\alpha h_\alpha = 1$, there exists an index β such that $h_\beta(p) > 0$. Hence $\mathrm{d}s^2(p) \ge h_\beta \mathrm{d}s_\beta^2(p)$. Thus $\mathrm{d}s^2$ is positive definite everywhere on M. □

Assume M is a generalized Riemannian manifold. When the local coordinate system is changed, the transformation formula for the components of a fundamental tensor G is given by

$$g'_{ij} = g_{kl} \frac{\partial u^k}{\partial u'^i} \frac{\partial u^l}{\partial u'^j}.$$

Since the matrix (g_{ij}) is nondegenerate, we may denote its inverse by (g^{ij}), i.e.,

$$g^{ik} g_{kj} = g_{jk} g^{ki} = \delta_j^i.$$

The transformation for g^{ij} under a change of coordinates is given by

$$g'^{ij} = g^{kl}\frac{\partial u'^i}{\partial u^k}\frac{\partial u'^j}{\partial u^l}.$$

Hence (g^{ij}) is a symmetric contravariant tensor of rank 2.

Using the fundamental tensor, we may identify a tangent space with a cotangent space, and hence a contravariant vector and a covariant vector can be viewed as different expressions of the same vector. In fact, if $X \in T_p(M)$, let

$$\alpha_X(Y) = G(X, Y), \quad Y \in T_p(M).$$

Then α_X is a linear functional on $T_p(M)$, i.e. $\alpha_X \in T_p^*(M)$. Conversely, since G is nondegenerate, any element of $T_p^*(M)$ can be expressed in the form α_X. Thus α establishes an isomorphism between $T_p(M)$ and $T_p^*(M)$. Componentwise, if

$$X = X^i\frac{\partial}{\partial u^i}, \quad \alpha_X = X_i \mathrm{d}u^i,$$

then we obtain from the relation of X and α_X that

$$X_i = g_{ij}X^j, \quad X^j = g^{ij}X_i.$$

In general, if $\left(t^{i_1\cdots i_r}_{j_1\cdots j_s}\right)$ is a (r, s)-type tensor, then

$$t^{i_1\cdots i_{r-1}}_{kj_1\cdots j_s} = g_{kl}t^{i_1\cdots i_{r-1}l}_{j_1\cdots j_s}, \quad t^{i_1\cdots i_r k}_{j_2\cdots j_s} = g^{kl}t^{i_1\cdots i_r}_{lj_2\cdots j_s}$$

are $(r-1, s+1)$-type and $(r+1, s-1)$-type tensors, respectively. These operations are usually called the **lowering** and **raising** of tensorial indices, respectively.

Definition 4.1.2 Suppose (M, G) is an m-dimensional generalized Riemannian manifold, and D is an affine connection on M. If

$$\mathrm{D}G = 0,$$

then D is called a **metric-compatible connection** on (M, G).

Condition $\mathrm{D}G = 0$ means that the fundamental tensor G is parallel with respect to metric-compatible connections. If the connection matrix of D under the local coordinates u^i is $\omega = (\omega_i^j)$, then

$$\mathrm{D}G = \left(\mathrm{d}g_{ij} - \omega_i^k g_{kj} - \omega_j^k g_{ik}\right) \otimes \mathrm{d}u^i \otimes \mathrm{d}u^j.$$

Thus $\mathrm{D}G = 0$ is equivalent to

$$dg_{ij} = \omega_i^k g_{kj} + \omega_j^k g_{ik},$$

or in matrix notation,

$$\mathrm{d}G = \omega \cdot G + G \cdot \omega^T,$$

where G represents the matrix

$$G = \begin{pmatrix} g_{11} & \cdots & g_{1m} \\ \vdots & & \vdots \\ g_{m1} & \cdots & g_{mm} \end{pmatrix}.$$

The geometric meaning of metric-compatible connections is that parallel translations preserve the metric. In particular, on a Riemannian manifold, the length of a tangent vector and the angle between two tangent vectors are invariant under parallel translations.

Theorem 4.1.2 (Fundamental Theorem of Riemanninan Geometry) Suppose M is an m-dimensional generalized Riemannian manifold. Then there exists a unique torsion-free and metric-compatible connection on M, called the **Levi-Civita connection** of M, or the **Riemannian connection** of M.

Proof. Suppose D is a torsion-free and metric-compatible connection on M. Denote the connection matrix of D under the local coordinates u^i by $\omega = (\omega_i^j)$, where

$$\omega_i^j = \Gamma_{ik}^j \mathrm{d}u^k.$$

Then we have

$$\begin{aligned} \mathrm{d}g_{ij} &= \omega_i^k g_{kj} + \omega_j^k g_{ki}, \\ \Gamma_{ik}^j &= \Gamma_{ki}^j. \end{aligned}$$

Denote

$$\Gamma_{ijk} = g_{lj}\Gamma_{ik}^j, \quad \omega_{ik} = g_{lk}\omega_i^l.$$

Then

$$\begin{aligned} \frac{\partial g_{ij}}{\partial u^k} &= \Gamma_{ijk} + \Gamma_{jik}, \\ \Gamma_{ijk} &= \Gamma_{kji}. \end{aligned}$$

Cycling the indices, we get

$$\frac{\partial g_{ik}}{\partial u^j} = \Gamma_{ikj} + \Gamma_{kij},$$
$$\frac{\partial g_{jk}}{\partial u^i} = \Gamma_{jki} + \Gamma_{kji}.$$

Therefore

$$\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k} = \Gamma_{ikj} + \Gamma_{kij} + \Gamma_{jki} + \Gamma_{kji} - \Gamma_{ijk} - \Gamma_{jik} = 2\Gamma_{ikj}.$$

We then obtain

$$\Gamma_{ikj} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}\right),$$

and then

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl}\left(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l}\right).$$

Thus the torsion-free and metric-compatible connection is determined uniquely by the metric tensor.

Conversely, the Γ_{ij}^k defined above indeed satisfy the transformation equation for connection coefficients under a change of local coordinates. Hence they define an affine connection D on M. Computations also verify that D is a torsion-free and metric-compatible connection on M. □

The Γ_{ikj} and Γ_{ij}^k defined above are called **Christoffel symbols** of the first kind and second kind, respectively.

It is more convenient to use an arbitrary frame field instead of the natural frame field in a neighborhood of a Riemannian manifold. A local frame field is a local section of the frame bundle. Suppose $(e_1, \cdots, e_m)$ is a local frame field with coframe field $(\theta^1, \cdots, \theta^m)$. Let

$$\mathrm{D}e_i = \theta_i^j e_j,$$

where $\theta = (\theta_i^j)$ is the connection matrix of D with respect to the frame field $(e_1, \cdots, e_m)$. Here the θ^i, θ_i^j are exactly the forms obtained by pulling the differential 1-forms θ^i and θ_i^j on the frame bundle P back to local sections. Hence by the structure equations, the statement that D is torsion-free is equivalent to the statement that the θ_i^j satisfy the equations

$$\mathrm{d}\theta^i - \theta^j \wedge \theta_j^i = 0.$$

If we still denote $g_{ij} = G(e_i, e_j)$, then the metric form is $\mathrm{d}s^2 = g_{ij}\theta^i\theta^j$. Since $G = g_{ij}\theta^i \otimes \theta^j$, we have

$$\mathrm{D}G = \left(\mathrm{d}g_{ij} - g_{ik}\theta_j^k - g_{kj}\theta_i^k\right) \otimes \theta^i \otimes \theta^j.$$

Therefore the condition for D to be metric-compatible is still

$$\mathrm{d}g_{ij} = g_{ik}\theta_j^k + g_{kj}\theta_i^k.$$

Now the Fundamental Theorem of Riemannian Geometry can be restated as follows.

Theorem 4.1.3 Suppose (M, G) is a generalized Riemannian manifold, and $\{\theta^i, 1 \leq i \leq m\}$ is a set of differential 1-forms on a neighborhood $U \subset M$ which is linearly independent everywhere. Then there exists a unique set of m^2 differential 1-forms θ_j^k on U such that

$$\mathrm{d}\theta^i - \theta^j \wedge \theta_j^i = 0,$$

and

$$\mathrm{d}g_{ij} = g_{ik}\theta_j^k + g_{kj}\theta_i^k,$$

where the g_{ij} are the components of G with respect to the local coframe field $\{\theta^i\}$, i.e. $G = g_{ij}\theta^i \otimes \theta^j$.

If M is a Riemannian manifold, and G is positive definite, then we can choose an orthogonal frame field $\{e_i, 1 \leq i \leq m\}$ in U with $g_{ij} = \delta_{ij}$, or equivalently,

$$\mathrm{d}s^2 = \sum_{i=1}^{m} (\theta^i)^2.$$

The condition for the connection to be metric-compatible then becomes

$$\theta_j^i + \theta_i^j = 0,$$

which implies that the connection matrix $\theta = (\theta_i^j)$ is skew-symmetric.

By definition, the curvature matrix of the Levi-Civita connection ω is

$$\Omega = \mathrm{d}\omega - \omega \wedge \omega.$$

Exterior differentiation of the equation

$$\mathrm{d}G = \omega \cdot G + G \cdot \omega^T$$

yields

$$d\omega \cdot G - \omega \wedge dG + dG \wedge \omega^T + G \cdot (d\omega)^T = 0,$$

and then

$$(d\omega - \omega \wedge \omega) \cdot G + G \cdot (d\omega - \omega \wedge \omega)^T = 0,$$

i.e.

$$\Omega \cdot G + (\Omega \cdot G)^T = 0.$$

Let

$$\Omega_{ij} = \Omega_i^k g_{kj},$$

then $\Omega \cdot G = (\Omega_{ij})$, and the above equation becomes

$$\Omega_{ij} + \Omega_{ji} = 0,$$

that is, Ω_{ij} is skew-symmetric with respect to the lower indices. By a direct calculation we get

$$\Omega_{ij} = d\omega_{ij} - \omega_i^k \wedge \omega_{jk}.$$

Also, we have

$$\Omega_i^j = \frac{1}{2} R_{ikl}^j du^k \wedge du^l,$$

where

$$R_{ikl}^j = \frac{\partial \Gamma_{il}^j}{\partial u^k} - \frac{\partial \Gamma_{ik}^j}{\partial u^l} + \Gamma_{il}^h \Gamma_{hk}^j - \Gamma_{ik}^h \Gamma_{hl}^j.$$

If we let

$$R_{ijkl} = R_{ikl}^h g_{hj},$$

then

$$\Omega_{ij} = \frac{1}{2} R_{ijkl} du^k \wedge du^l,$$

and

$$R_{ijkl} = \frac{\partial \Gamma_{ijl}}{\partial u^k} - \frac{\partial \Gamma_{ijk}}{\partial u^l} + \Gamma_{ik}^h \Gamma_{jhl} - \Gamma_{il}^h \Gamma_{jhk}.$$

Here R_{ijkl} is a covariant tensor of rank 4. It is determined completely by a given generalized Riemannian metric on M, and is called the **curvature tensor** of the generalized Riemannian manifold M.

Theorem 4.1.4 The curvature tensor R_{ijkl} of a generalized Riemannian manifold satisfies the following properties:

1. $R_{ijkl} = -R_{jikl} = -R_{ijlk}$;
2. $R_{ijkl} + R_{iklj} + R_{iljk} = 0$;

3. $R_{ijkl} = R_{klij}$.

Proof. The skew-symmetry of R^j_{ikl} in the last two lower indices implies the same property of R_{ijkl}, i.e.,

$$R_{ijkl} = -R_{ijlk}.$$

Since we have

$$0 = \Omega_{ij} + \Omega_{ji} = \frac{1}{2}(R_{ijkl} + R_{jikl})\mathrm{d}u^k \wedge \mathrm{d}u^l,$$

it must be true that

$$R_{ijkl} + R_{jikl} = 0.$$

From the torsion-free property of the Levi-Civita connection we have

$$\mathrm{d}u^i \wedge \omega_{ij} = 0.$$

Exteriorly differentiating this and using the formula

$$\Omega_{ij} = \mathrm{d}\omega_{ij} + \omega_i^k \wedge \omega_{jk},$$

we then have

$$\mathrm{d}u^i \wedge (\Omega_{ij} - \omega_i^k \wedge \omega_{jk}) = 0,$$

thus

$$\mathrm{d}u^i \wedge \Omega_{ij} = 0.$$

Therefore

$$R_{jikl}\mathrm{d}u^i \wedge \mathrm{d}u^k \wedge \mathrm{d}u^l = 0,$$

or equivalently,

$$(R_{jikl} + R_{jkli} + R_{jlik})\mathrm{d}u^i \wedge \mathrm{d}u^k \wedge \mathrm{d}u^l = 0.$$

Since the coefficients are skew-symmetric in the last three indices, we have

$$R_{jikl} + R_{jkli} + R_{jlik} = 0.$$

We can cycle the indices to obtain

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0.$$

It follows that

$$\begin{aligned} 0 &= (R_{ijkl} + R_{iklj} + R_{iljk}) - (R_{jikl} + R_{jkli} + R_{jlik}) \\ &= 2R_{ijkl} + R_{iklj} + R_{iljk} + R_{jkil} + R_{ljik}. \end{aligned}$$

Similarly we also have

$$2R_{klij} + R_{kijl} + R_{kjli} + R_{likj} + R_{jlki} = 0.$$

Due to the skew-symmetry property 1, we finally have

$$R_{ijkl} = R_{klij}.$$

□

As a corollary, under the same conditions as in Theorem 4.1.4, we have

$$R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0.$$

Further, from $\mathrm{D}G = 0$ we have

$$g_{ij,k} = 0,$$

and hence

$$R_{ijkl,h} = (g_{jp}R^p_{ikl})_{,h} = g_{jp}R^p_{ikl,h}.$$

Thus it follows from

$$R^j_{ikl,h} + R^j_{ilh,k} + R^j_{ihk,l} = 0$$

that

$$R_{ijkl,h} + R_{ijlh,k} + R_{ijhk,l} = 0.$$

This is also called the **Bianchi identity**.

4.2 Geodesic Normal Coordinates

Definition 4.2.1 Suppose M is an m-dimensional Riemannian manifold. If a parametrized curve C is a geodesic curve in M with respect to the Levi-Civita connection, then C is called a **geodesic** of the Riemannian manifold M.

Suppose the coefficients of the Levi-Civita connection D under the local coordinates u^i are Γ^i_{jk}. Then the curve $C : u^i = u^i(t), 1 \leq i \leq m$ is a geodesic if it satisfies the system of second order differential equations

$$\frac{\mathrm{d}^2u^i}{\mathrm{d}t^2} + \Gamma^i_{jk}\frac{\mathrm{d}u^j}{\mathrm{d}t}\frac{\mathrm{d}u^k}{\mathrm{d}t} = 0, \quad 1 \leq i \leq m.$$

By definition, the tangent vector of a geodesic is parallel along the curve with respect to the Levi-Civita connection, which also preserves metric properties under parallel displacement. Therefore the length of the tangent vector

$$X = X^i \frac{\partial}{\partial u^i} = \frac{\mathrm{d}u^i}{\mathrm{d}t} \frac{\partial}{\partial u^i}$$

of a geodesic is constant, that is,

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \text{const.}$$

Hence we see that the parameter for a geodesic curve in a Riemannian manifold must be a linear function of the arc length s, i.e.

$$t = \lambda s + \mu,$$

where $\lambda \neq 0$ and μ are constants.

The discussions below only assume that M is an affine connection space. Suppose the equation of a geodesic under the coordinate system $(U; u^i)$ is given by

$$\frac{\mathrm{d}^2 u^i}{\mathrm{d}t^2} + \Gamma^i_{jk} \frac{\mathrm{d}u^j}{\mathrm{d}t} \frac{\mathrm{d}u^k}{\mathrm{d}t} = 0, \quad 1 \leq i \leq m.$$

By the theory of ordinary differential equations, there exist for any point $x_0 \in U$ a neighborhood $W \subset U$ of x_0 and positive numbers r, δ such that for any initial value $x \in W$ and $\alpha \in \mathbb{R}^m$ satisfying $\|\alpha\| < r$, the system of equations has a unique solution in U expressed as

$$u^i = f^i(t, x^k, \alpha^k), \quad |t| < \delta,$$

that satisfies the initial conditions

$$u^i(0) = f^i(0, x^k, \alpha^k) = x^i,$$
$$\frac{\mathrm{d}u^i}{\mathrm{d}t}(0) = \left. \frac{\partial f^i(t, x^k, \alpha^k)}{\partial t} \right|_{t=0} = \alpha^i.$$

Furthermore, the functions f^i depend smoothly on the independent variable t and the initial values x^k, α^k.

If we choose a nonzero constant c, then the functions $f^i(ct, x^k, \alpha^k), x \in W, \|\alpha\| < r, |t| < \delta/|c|$ still satisfy the system of equations with initial values

$$\left. f^i(ct, x^k, \alpha^k) \right|_{t=0} = x^i,$$
$$\left. \frac{\partial f^i(ct, x^k, \alpha^k)}{\partial t} \right|_{t=0} = c\alpha^k.$$

By the uniqueness property of the solution of the system of differential equations, whenever $\|\alpha\|, \|c\alpha\| < r$ and $|t|, |ct| < \delta$, we have

$$f^i(ct, x^k, \alpha^k) = f^i(t, x^k, c\alpha^k).$$

Since the left hand side of the above equation is always defined when $x \in W, \|\alpha\| < r, |t| < \delta/|c|$, we can use it to define the right hand side. Thus the function $f^i(t, x^k, \alpha^k)$ is always defined for $x \in W, |t| < \delta/|c|, \|\alpha\| < |c|r$. In particular, we can choose $|c| < \delta$, so that $f^i(t, x^k, \alpha^k)$ is defined for $x \in W, |t| \leq 1$ and $\|\alpha\| < |c|r$. Let

$$u^i = f^i(1, x^k, \alpha^k),$$

then

$$f^i(1, x^k, 0) = f^i(0, x^k, \alpha^k) = x^k.$$

Thus for a fixed $x \in W$, this provides a smooth map from a neighborhood of the origin in the tangent space $T_x(M)$ to a neighborhood of x in the manifold M. Because

$$\alpha^i = \left.\frac{\partial f^i(t, x^k, \alpha^k)}{\partial t}\right|_{t=0} = \left.\frac{\partial f^i(1, x^k, t\alpha^k)}{\partial t}\right|_{t=0} = \left.\frac{\partial f^i(1, x^k, \alpha^k)}{\partial \alpha^j}\right|_{\alpha=0} \cdot \alpha^j,$$

we have

$$\left(\frac{\partial u^i}{\partial \alpha^j}\right)_{\alpha=0} = \delta^i_j.$$

Hence the α^i can be chosen to be local coordinates of x in M, called the **geodesic normal coordinates** of x, or simply **normal coordinates**. A normal coordinate system of a point in M is determined up to a nondegenerate linear transformation.

Fix $\alpha^k = \alpha^k_0$. As t changes, $t\alpha^k_0$ describes a straight line in $T_x(M)$ starting from the origin, and traces a geodesic curve on the manifold starting from x and tangent to the tangent vector (α^k_0). Therefore the equation for this geodesic curve under the normal coordinate system α^i is

$$\alpha^k = t\alpha^k_0,$$

where α^k_0 is a constant.

Theorem 4.2.1 If M is a torsion-free affine connection space, then with respect to a normal coordinate system α^i at the point x, the connection coefficients Γ^i_{jk} are zero at x.

Proof. Since the geodesic curve $\alpha^i = t\alpha_0^i$ satisfies the system of differential equations for geodesics under the normal coordinate system α^i, we have for any α_0^k,

$$\Gamma^i_{jk}\alpha_0^j\alpha_0^k = 0.$$

Since Γ^i_{jk} is symmetric in the lower indices for torsion-free connections, we have

$$\Gamma^i_{jk}(0) = 0, \quad 1 \le i, j, k \le m.$$

□

Theorem 4.2.2 For any point x_0 in an affine connection space M, there exists a neighborhood W of x_0 such that every point in W has a normal coordinate neighborhood that contains W.

Proof. Suppose $(U; u^i)$ is a normal coordinate system at a point x_0. Let

$$U(x_0; \rho) = \left\{ x \in U \;\middle|\; \sum_{i=1}^{m} (u^i(x))^2 < \rho^2 \right\}.$$

By the above discussion, there exists a neighborhood $W = U(x_0; r)$ of x_0 and a positive number δ such that for any $x \in W$ and $\alpha \in \mathbb{R}^m, \|\alpha\| < \delta$, there is a unique geodesic curve

$$u^i = f^i(t, x^k, \alpha^k), \quad |t| < 2,$$

with initial condition (x, α). Let

$$B(0; \delta) = \{\alpha \in \mathbb{R}^m \mid \|\alpha\| < \delta\}.$$

Then we have a map $\varphi : W \times B(0; \delta) \to W \times U$ such that

$$\varphi(x, \alpha) = (x^k, f^k(1, x^i, \alpha^i)), \quad x \in W, \alpha \in B(0; \delta).$$

The map φ is smooth since the function f^k depend on x and α smoothly. Noting that

$$\left.\frac{\partial(x^k, f^k)}{\partial(x^i, \alpha^i)}\right|_{(x_0,0)} = 1,$$

the Jacobian matrix of the map φ is nondegenerate near the point $(x_0, 0) \in W \times B(0; \delta)$. By the Inverse Function Theorem, there exists a neighborhood V of the point $(x_0, 0)$ and a positive number $a < \delta$ such that $\varphi : V \to U(x_0; a) \times U(x_0; a)$ is a diffeomorphism. For any $x \in U(x_0; a)$, let

$$V_x = \{\alpha \in B(0; a) \mid (x, \alpha) \in V\}.$$

Then the map

$$u^i = f^i(1, x^k, \alpha^k), \quad \alpha \in V_x$$

is a diffeomorphism from V_x to $U(x_0; a)$. Choose $W' = U(x_0; a)$, and then the above formula shows that W' has the desired property. □

Corollary 4.2.3 For every point x_0 in an affine connection space M, there exists a neighborhood W of x_0 such that any two points in W can be connected by a geodesic curve.

Theorem 4.2.4 A torsion-free affine connection is completely determined locally by the curvature tensor.

Proof. Consider a normal coordinate system α^i at a fixed point O. Choose a natural frame at O, and parallel displace the frame along the geodesic curves starting from O. Thus we get a frame field $\{e_i, 1 \le i \le m\}$ in a neighborhood of O. Let θ^i be the dual differential 1-forms of e_j, and denote the restriction of the everywhere linearly independent m^2 differential 1-forms θ_i^j of the frame bundle to the above frame field by the same notation. Then θ^i, θ_i^j are differential 1-forms of t, α^k. When the α^k are constants, θ^i, θ_i^j are restricted to the geodesic curve $\alpha^i t$. Since the frame field is parallel along the geodesic curve $\alpha^i t$, we have

$$\begin{aligned} \theta^i &= \alpha^i \mathrm{d}t + \bar{\theta}^i, \\ \theta_i^j &= \bar{\theta}_i^j, \end{aligned}$$

where $\bar{\theta}^i$ and $\bar{\theta}_i^j$ are the parts of θ^i and θ_i^j without $\mathrm{d}t$. Plugging this into the structure equations and comparing the terms with $\mathrm{d}t$, we obtain

$$\begin{aligned} \frac{\partial \bar{\theta}^i}{\partial t} &= \mathrm{d}\alpha^i + \alpha^j \bar{\theta}_j^i, \\ \frac{\partial \bar{\theta}_i^j}{\partial t} &= \alpha^k S_{ikl}^j \bar{\theta}^l. \end{aligned}$$

Differentiating the first formula with respect to t again, we obtain

$$\frac{\partial^2 \bar{\theta}^i}{\partial t^2} = \alpha^j \frac{\partial \bar{\theta}_j^i}{\partial t} = \alpha^j \alpha^k S_{jkl}^i \bar{\theta}^l.$$

Since the frame field e_i is parallel along any direction at the point O, we have

$$\bar{\theta}_i^j \Big|_{t=0} = 0,$$

and then

$$\left.\frac{\partial\bar{\theta}^i}{\partial t}\right|_{t=0} = \mathrm{d}\alpha^i.$$

Moreover, by definition we have

$$\left.\theta^i\right|_{t=0} = \alpha^i\mathrm{d}t,$$

and thus

$$\left.\bar{\theta}^i\right|_{t=0} = 0.$$

For a given curvature tensor, the system of second-order ordinary differential equations

$$\frac{\partial^2\bar{\theta}^i}{\partial t^2} = \alpha^j\alpha^k S^i_{jkl}\bar{\theta}^l$$

has a unique solution for $\bar{\theta}^i$ under the initial conditions, and $\bar{\theta}^j_i$ is then completely determined. Hence the curvature tensor completely determines the torsion-free affine connection locally. □

Now assume M is an m-dimensional Riemannian manifold. Suppose $x_0 \in M$, and choose a fixed orthogonal frame F_0 in the tangent space $T_{x_0}(M)$. Then the normal coordinate system u^i at x_0 can be expressed as $u^i = \alpha^i s$, where (α^i) is a unit vector in $T_{x_0}(M)$ and s is the arc length of the geodesic curves starting from x_0. Displace the frame F_0 parallel along the geodesic curves originating from x_0 to obatin an orthogonal frame field in a neighborhood of x_0. We can write

$$\theta^i = \alpha^i\mathrm{d}s + \bar{\theta}^i, \quad \theta^j_i = \bar{\theta}^j_i,$$

where $\bar{\theta}^i, \bar{\theta}^j_i$ do not contain the differential $\mathrm{d}s$, and satisfy the equations

$$\begin{aligned}
\frac{\partial\bar{\theta}^i}{\partial s} &= \mathrm{d}\alpha^i + \alpha^j\bar{\theta}^i_j,\\
\frac{\partial\bar{\theta}^j_i}{\partial s} &= \alpha^k S^j_{ikl}\bar{\theta}^l,\\
\bar{\theta}^j_i + \bar{\theta}^i_j &= 0,
\end{aligned}$$

with initial conditions

$$\left.\bar{\theta}^i\right|_{s=0} = 0, \quad \left.\bar{\theta}^j_i\right|_{s=0} = 0, \quad \left.\frac{\partial\bar{\theta}^i}{\partial s}\right|_{s=0} = \mathrm{d}\alpha^i.$$

The arc length element near the point O can be expressed by

$$\mathrm{d}\sigma^2 = \sum_{i=1}^{m}(\theta^i)^2 = \mathrm{d}s^2 + 2\mathrm{d}s\sum_{i=1}^{m}\alpha^i\bar{\theta}^i + \sum_{i=1}^{m}(\bar{\theta}^i)^2.$$

Since

$$\sum_{i=1}^{m}(\alpha^i)^2 = 1,$$

we have

$$\sum_{i=1}^{m}\alpha^i\mathrm{d}\alpha^i = 0.$$

Together with

$$\bar{\theta}_i^j + \bar{\theta}_j^i = 0,$$

we see that

$$\frac{\partial}{\partial s}\left(\sum_{i=1}^{m}\alpha^i\bar{\theta}^i\right) = \sum_{i=1}^{m}\alpha^i\left(\mathrm{d}\alpha^i + \sum_{j=1}^{m}\alpha^j\bar{\theta}_j^i\right) = 0.$$

Therefore

$$\sum_{i=1}^{m}\alpha^i\bar{\theta}^i = \sum_{i=1}^{m}\alpha^i\bar{\theta}^i\bigg|_{s=0} = 0.$$

Hence the arc length element near O is

$$\mathrm{d}\sigma^2 = \mathrm{d}s^2 + \sum_{i=1}^{m}(\theta^i)^2.$$

Theorem 4.2.5 For every point O in a Riemannian manifold M, there exists a normal coordinate neighborhood W such that

1. Every point in W has a normal coordinate neighborhood that contains W.
2. The geodesic curve that connects O and $p \in W$ is the unique shortest curve in W connecting these two points.

Proof. Applying Theorem 4.2.2 to the Levi-Civita connection of M, and condition 1 follows. Now assume that u^i is the normal coordinate system of the point O given by $u^i = \alpha^i s$. A normal coordinate neighborhood W as required in consition 1 is

$$W = \left\{ p \in M \,\middle|\, \sum_{i=1}^{m} (u^i(p))^2 < \varepsilon^2 \right\},$$

where ε is a sufficiently small positive number. Because W is a normal coordinate neighborhood, for any $p \in W$, there exists a unique geodesic curve γ in W that connects O and p. Suppose the length of γ is s_0.

Suppose C is any piecewise smooth curve in W that connects O and p. We may assume that the parametrized equation for C is $u^i = u^i(s)$, where s is the arc length parameter of γ. Then the arc length of C is

$$\int_0^{s_0} \mathrm{d}\sigma = \int_0^{s_0} \sqrt{\mathrm{d}s^2 + \sum_{i=1}^{m} (\theta^i)^2} \geq \int_0^{s_0} \mathrm{d}s = s_0.$$

If C is the shortest path in W connecting O and p, then the equality holds. Hence we must have

$$\bar{\theta}^i = 0$$

along the curve C. If we write

$$\bar{\theta}^i = s\mathrm{d}\alpha^i + A^i_j \mathrm{d}\alpha^j,$$

then the A^i_j satisfy the initial conditions

$$A^i_j\Big|_{s=0} = 0, \qquad \frac{\partial A^i_j}{\partial s}\Bigg|_{s=0} = 0.$$

This implies that $A^i_j = o(s)$ when $s \to 0$. Since

$$\mathrm{d}\alpha^i + \frac{A^i_j}{s}\mathrm{d}\alpha^j = 0$$

holds on C, we can let $s \to 0$ to obtain

$$\mathrm{d}\alpha^i = 0, \quad \alpha^i = \text{const}.$$

It follows that C is a geodesic curve connecting O and p, i.e. $C = \gamma$. □

Theorem 4.2.6 Suppose U is a normal coordinate neighborhood of the point O. Then there exists a positive number ε such that, for any $0 < \delta < \varepsilon$, the hypersphere

$$\Sigma_\delta = \left\{ p \in U \,\middle|\, \sum_{i=1}^{m} (u^i(p))^2 = \delta^2 \right\}$$

has the following properties:

1. Every point on Σ_δ can be connected to O by a unique shortest geodesic curve in U.

2. Any geodesic curve tangent to Σ_δ is strictly outside Σ_δ in a deleted neighborhood of the tangent point.

Proof. Choose W to be a normal coordinate neighborhood as required in Theorem 4.2.5. We may assume that W is a spherical neighborhood with radius ε. When $0 < \delta < \varepsilon$, since $\Sigma_\delta \subset W \subset U$ and U is a normal coordinate neighborhood, property 1 is just a corollary of Theorem 4.2.5.

The equation of Σ_δ can be written as

$$F(u^1, \cdots, u^m) = \frac{1}{2}[(u^1)^2 + \cdots + (u^m)^2 - \delta^2] = 0.$$

Suppose γ is a geodesic curve tangent to Σ_δ at p, and its equation is

$$u^i = u^i(\sigma),$$

where σ is the arc length of γ measured from the point p. Then

$$F(u^i(\sigma))|_{\sigma=0} = 0.$$

By the discussion before Theorem 4.2.5, the hypersphere Σ_δ is orthogonal to geodesic curves starting from the point O, thus the geodesic curve γ tangent to Σ_δ at the point p should be orthogonal to the geodesic curve connecting O and p. Therefore

$$\sum_{i=1}^{m} u^i(\sigma) \frac{\mathrm{d}u^i}{\mathrm{d}\sigma} \bigg|_{\sigma=0} = 0.$$

Direct calculation yields

$$\frac{\mathrm{d}}{\mathrm{d}\sigma} F(u^i(\sigma)) \bigg|_{\sigma=0} = \sum_{i=1}^{m} u^i(\sigma) \frac{\mathrm{d}u^i}{\mathrm{d}\sigma} \bigg|_{\sigma=0} = 0,$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}\sigma^2}F(u^i(\sigma))\bigg|_{\sigma=0} = \sum_{i,j=1}^{m}\left[\delta_{ij} - \sum_{k=1}^{m} u^k(p)\Gamma_{ij}^k(p)\right] \cdot \frac{\mathrm{d}u^i}{\mathrm{d}\sigma}\bigg|_{s=0} \cdot \frac{\mathrm{d}u^j}{\mathrm{d}\sigma}\bigg|_{s=0}.$$

Since $(U; u^i)$ is a normal coordinate system, we have

$$\Gamma_{ij}^k(p) = 0.$$

Hence we can choose a sufficiently small $\varepsilon > 0$ such that whenever $0 < \delta < \varepsilon$, the second-order derivative of $F(u^i(\sigma))$ with respect to σ at $\sigma = 0$ is always positive. Thus $F(u^i(\sigma))$ is strictly positive near p, which means that the geodesic curve lies strictly outside Σ_δ near p, and has only one point in common with Σ_δ, namely p. □

Definition 4.2.2 Suppose M is a connected Riemannian manifold, and p, q are two arbitrary points in M. Let

$$\rho(p, q) = \inf \widehat{pq},$$

where $\widehat{pq}$ denotes the arc length of a curve connecting p and q with measurable arc length. Then $\rho(p, q)$ is called the **distance** between points p and q.

Theorem 4.2.7 The function $\rho : M \times M \to \mathbb{R}$ is a metric on M and makes M a metric space. The topology of M as a metric space and the original topology of M as a manifold are equivalent.

Theorem 4.2.8 There exists a η-ball neighborhood W at any point p in a Riemannian manifold M, where η is a sufficiently small positive number, such that any two points in W can be connected by a geodesic curve inside W. Such a neighborhood is called a **geodesic convex neighborhood**.

Proof. Suppose $p \in M$. There exists a ball-shaped normal coordinate neighborhood U of p with radius ε such that for any point q in U there is a normal coordinate neighborhood V_q that contains U. We may assume that ε also satisfies the requirements of Theorem 4.2.6. Choose a positive $\eta < \varepsilon/4$. We will show that the η-ball neighborhood W of p is a geodesic convex neighborhood of p.

Choose any $q_1, q_2 \in W$. Then

$$\rho(q_1, q_2) \leq \rho(p, q_1) + \rho(p, q_2) < 2\eta \leq \frac{\varepsilon}{2}.$$

Suppose $U(q_1;\varepsilon/2)$ is an $\varepsilon/2$-ball neighborhood of q_1, then $q_2 \in U(q_1;\varepsilon/2) \subset U \subset V_{q_1}$. By Theorem 4.2.5, there exists a unique geodesic curve γ in $U(q_1;\varepsilon/2)$ connecting q_1 and q_2, whose length is precisely $\rho(q_1, q_2)$. We prove that the geodesic curve γ lies inside W. Since $\gamma \subset U(q_1;\varepsilon/2) \subset U$, the function $\rho(p,q), q \in \gamma$ is bounded. If γ does not lie inside W completely, then the function $\rho(p,q), q \in \gamma$ must attain its maximum at an interior point q_0 of γ. Let $\delta = \rho(p, q_0)$. Then $\delta < \varepsilon$, and the hypersphere Σ_δ is tangent to γ at q_0. By Theorem 4.2.6, γ lies completely outside Σ_δ near q_0, contradicting the fact that $\rho(p,q), q \in \gamma$ attains its maximum at q_0. Therefore $\gamma \subset W$. $\square$

4.3 Sectional Curvature

Suppose M is an m-dimensional Riemannian manifold whose curvature tensor R is a covariant tensor of rank 4, and u^i is a local coordinate system in M. Then R can be expressed as

$$R = R_{ijkl} \mathrm{d}u^i \otimes \mathrm{d}u^j \otimes \mathrm{d}u^k \otimes \mathrm{d}u^l.$$

At every point $p \in M$, we have a multilinear function $R : T_p(M) \times T_p(M) \times T_p(M) \times T_p(M) \to \mathbb{R}$, defined by

$$R(X,Y,Z,W) = \langle X \otimes Y \otimes Z \otimes W, R\rangle.$$

If we let

$$X = X^i \frac{\partial}{\partial u^i}, \quad Y = Y^i \frac{\partial}{\partial u^i}, \quad Z = Z^i \frac{\partial}{\partial u^i}, \quad W = W^i \frac{\partial}{\partial u^i},$$

then

$$R(X,Y,Z,W) = R_{ijkl} X^i Y^j Z^k W^l.$$

In particular

$$R_{ijkl} = R\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}, \frac{\partial}{\partial u^k}, \frac{\partial}{\partial u^l}\right).$$

We have already interpreted the curvature tensor of a connection D as a curvature operator: for any given $Z, W \in T_p(M)$, $R(Z,W)$ is a linear map from $T_p(M)$ to $T_p(M)$ defined by

$$R(Z,W)X = R^j_{ikl} X^i Z^k W^l \frac{\partial}{\partial u^j}.$$

If D is the Levi-Civita connection of a Riemannian manifold M, then we have

$$R(X,Y,Z,W) = R(Z,W)X \cdot Y,$$

where $\cdot$ on the right hand side is the inner product defined by

$$X \cdot Y = G(X, Y).$$

By the properties of R_{ijkl}, the 4-linear function $R(X, Y, Z, W)$ has the following properties:

1. $R(X, Y, Z, W) = -R(X, Y, W, Z) = -R(Y, X, Z, W)$;
2. $R(X, Y, Z, W) + R(X, Z, W, Y) + R(X, W, Y, Z) = 0$;
3. $R(X, Y, Z, W) = R(Z, W, X, Y)$.

Using the fundamental tensor G of M, we can also define a function

$$G(X, Y, Z, W) = G(X, Z)G(Y, W) - G(X, W)G(Y, Z).$$

Obviously the function defined above is linear with respect to every variable, and also has the same properties 1-3 as $R(X, Y, Z, W)$. If $X, Y \in T_p(M)$, then

$$G(X, Y, X, Y) = |X|^2|Y|^2 - (X \cdot Y)^2 = |X|^2|Y|^2 \sin^2 \angle(X, Y).$$

Therefore when X, Y are linearly independent, $G(X, Y, X, Y)$ is precisely the square of the area of the parallelogram determined by the tangent vectors X and Y. Hence $G(X, Y, X, Y) \neq 0$.

Suppose X', Y' are another two linearly independent tangent vectors at the point p, and that they span the same 2-dimensional tangent subspace E as that spanned by X and Y. Then we may assume that

$$X' = aX + bY, \quad Y' = cX + dY,$$

where $ad - bc \neq 0$. By properties 1-3 we have

$$R(X', Y', X', Y') = (ad - bc)^2 R(X, Y, X, Y),$$
$$G(X', Y', X', Y') = (ad - bc)^2 G(X, Y, X, Y).$$

Thus

$$\frac{R(X', Y', X', Y')}{G(X', Y', X', Y')} = \frac{R(X, Y, X, Y)}{G(X, Y, X, Y)}.$$

This implies that the above expression is a function of the 2-dimensional subspace E of $T_p(M)$, and is independent of the choice of X and Y.

Definition 4.3.1 Suppose E is a 2-dimensional subspace of $T_p(M)$, and X, Y are any two linearly independent vectors in E. Then

$$K(E) = -\frac{R(X,Y,X,Y)}{G(X,Y,X,Y)}$$

is a function of E independent of the choice of X and Y in E. It is called the **Riemannian curvature**, or **sectional curvature**, of M at (p, E).

The product of the two principal curvatures at a point on a surface in 3-dimensional Euclidean space is called the **total curvature**, or **Gauss curvature**, of the surface at that point. The **Theorema Egregium** shows that the total curvature K depends only on the first fundamental form of the surface as

$$K = -\frac{R_{1212}}{g},$$

where

$$g = g_{11}g_{22} - g_{12}^2$$

and

$$R_{1212} = \frac{\partial \Gamma_{122}}{\partial u^1} - \frac{\partial \Gamma_{121}}{\partial u^2} + \Gamma_{11}^h \Gamma_{2h2} - \Gamma_{12}^h \Gamma_{2h1}.$$

Suppose $m \geq 3$ and E is a 2-dimensional subspace of $T_p(M)$. Choose an orthogonal frame $\{e_i\}$ at p such that E is spanned by $\{e_1, e_2\}$. Suppose u^i is the geodesic normal coordinate system determined by this frame near p. Now consider the 2-dimensional submanifold S of all geodesic curves starting from p and tangent to E. Then the equation for S is

$$u^r = 0, \quad 3 \leq r \leq m,$$

and (u^1, u^2) are the normal coordinates of S at p. S is called the **geodesic submanifold** at p tangent to E. We will prove that the sectional curvature $K(E)$ of M at (p, E) is exactly the total curvature of the surface S, with Riemannian metric induced from M, at p.

Suppose the Riemannian metric of M near p is

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}u^i \mathrm{d}u^j.$$

Then its induced metric on S is

$$\mathrm{d}\bar{s}^2 = \bar{g}_{\alpha\beta}\mathrm{d}u^\alpha \mathrm{d}u^\beta, \quad 1 \leq \alpha, \beta \leq 2,$$

where

$$\bar{g}_{\alpha\beta}(u^1, u^2) = g_{\alpha\beta}(u^1, u^2, 0, \cdots, 0).$$

Therefore

$$\Gamma_{\alpha\beta\gamma}|_S = \frac{1}{2}\left(\frac{\partial g_{\beta\gamma}}{\partial u^\alpha} + \frac{\partial g_{\alpha\beta}}{\partial u^\gamma} - \frac{\partial g_{\alpha\gamma}}{\partial u^\beta}\right)\Bigg|_S$$
$$= \frac{1}{2}\left(\frac{\partial \bar{g}_{\beta\gamma}}{\partial u^\alpha} + \frac{\partial \bar{g}_{\alpha\beta}}{\partial u^\gamma} - \frac{\partial \bar{g}_{\alpha\gamma}}{\partial u^\beta}\right) = \overline{\Gamma}_{\alpha\beta\gamma}.$$

Since (u^i) and (u^α) are normal coordinate systems of M and S, respectively, at p, we have

$$\overline{\Gamma}_{\alpha\beta\gamma}(p) = \Gamma_{ijk}(p) = 0.$$

Hence

$$R_{1212}(p) = \left(\frac{\partial \Gamma_{122}}{\partial u^1} - \frac{\partial \Gamma_{121}}{\partial u^2} + \Gamma^h_{11}\Gamma_{2h2} - \Gamma^h_{12}\Gamma_{2h1}\right)_p$$
$$= \left(\frac{\partial \overline{\Gamma}_{122}}{\partial u^1} - \frac{\partial \overline{\Gamma}_{121}}{\partial u^2}\right)_p = \overline{R}_{1212}(p).$$

The sectional curvature of M at (p, E) is then

$$K(E) = -\frac{R(e_1, e_2, e_1, e_2)}{G(e_1, e_2, e_1, e_2)} = -\left.\frac{R_{1212}}{g_{11}g_{22} - g_{12}^2}\right|_p = -\left.\frac{\overline{R}_{1212}}{\bar{g}_{11}\bar{g}_{22} - \bar{g}_{12}^2}\right|_p = \overline{K}(p).$$

The right hand side is precisely the total curvature of the surface S at p.

Theorem 4.3.1 The curvature tensor of a Riemannian manifold M at a point p is uniquely determined by the sectional curvatures of all the 2-dimensional tangent subspaces at p.

Proof. Suppose there is a 4-linear function $\overline{R}(X, Y, Z, W)$ satisfying all the properties 1-3 of the curvature tensor $R(X, Y, Z, W)$, and that for any two linearly independent tangent vectors X, Y at p,

$$\frac{\overline{R}(X, Y, X, Y)}{G(X, Y, X, Y)} = \frac{R(X, Y, X, Y)}{G(X, Y, X, Y)}.$$

We will show that for any $X, Y, Z, W \in T_p(M)$, we have

$$\overline{R}(X, Y, Z, W) = R(X, Y, Z, W).$$

If we let

$$S(X, Y, Z, W) = \overline{R}(X, Y, Z, W) - R(X, Y, Z, W),$$

Then S is also a 4-linear function satisfying the properties 1-3 and for any $X, Y \in T_p(M)$, it holds that

$$S(X, Y, X, Y) = 0.$$

It suffices to show that S is the zero function.

First we have

$$S(X + Z, Y, X + Z, Y) = 0.$$

Expanding this and using the properties of S we obtain

$$S(X, Y, Z, Y) = 0.$$

Thus

$$S(X, Y + W, Z, Y + W) = 0,$$

and by expanding we obtain

$$S(X, Y, Z, W) + S(X, W, Z, Y) = 0.$$

Therefore

$$S(X, Y, Z, W) = -S(X, W, Z, Y) = S(X, W, Y, Z).$$

A similar argument shows that

$$S(X, Y, Z, W) = S(X, W, Y, Z) = S(X, Z, W, Y).$$

On the other hand, it holds the identity

$$S(X, Y, Z, W) + S(X, Z, W, Y) + S(X, W, Y, Z) = 0.$$

Thus

$$S(X, Y, Z, W) = 0$$

and the proof is completed. □

Definition 4.3.2 Suppose M is a Riemannian manifold. If the sectional curvature $K(E)$ at the point p is a constant, i.e. independent of E, then we say that M is **wandering** at p.

If M is wandering at p, then the sectional curvature of M at p can be denoted by $K(p)$. Hence for any $X, Y \in T_p(M)$ we have

$$R(X, Y, X, Y) = -K(p)G(X, Y, X, Y).$$

According to the proof of Theorem 4.3.1, for any $X, Y, Z, W \in T_p(M)$, we have

$$R(X, Y, Z, W) = -K(p)G(X, Y, Z, W).$$

Thus the condition for a Riemannian manifold to be wandering at p is

$$R_{ijkl}(p) = -K(p)(g_{ik}g_{jl} - g_{il}g_{jk})(p),$$

or

$$\Omega_{ij}(p) = -K(p) \cdot \theta_i \wedge \theta_j(p),$$

where $\theta_i = g_{ij}\mathrm{d}u^j$.

Definition 4.3.3 If M is a Riemannian manifold which is wandering at every point and the sectional curvature $K(p)$ is a constant function on M, then M is called a **constant curvature space**.

Theorem 4.3.2 (F. Schur's Theorem) Suppose M is a connected m-dimensional Riemannian manifold that is everywhere wandering. If $m \geq 3$, then M is a constant curvature space.

Proof. Since M is wandering everywhere, it holds that

$$\Omega_{ij} = -K\theta_i \wedge \theta_j,$$

where K is a smooth function on M, and $\theta_i = g_{ij}\mathrm{d}u^j$. Exterior differentiation yields

$$\mathrm{d}\Omega_{ij} = -\mathrm{d}K \wedge \theta_i \wedge \theta_j - K\mathrm{d}\theta_i \wedge \theta_j + K\theta_i \wedge \mathrm{d}\theta_j.$$

However,

$$\mathrm{d}\theta_i = \mathrm{d}g_{ij} \wedge \mathrm{d}u^j = (g_{ik}\omega_j^k + g_{kj}\omega_i^k) \wedge \mathrm{d}u^j = (\omega_{ij} + \omega_{ji}) \wedge \mathrm{d}u^j,$$

where

$$\omega_{ij} = g_{jk}\omega_i^k = \Gamma_{ijk}\mathrm{d}u^k.$$

Since the Levi-Civita connection is torsion-free, we have

$$\omega_{ji} \wedge \mathrm{d}u^j = \Gamma_{jik}\mathrm{d}u^k \wedge \mathrm{d}u^j = 0,$$

and hence

$$\mathrm{d}\theta_i = \omega_{ij} \wedge \mathrm{d}u^j = \omega_i^j \wedge \theta_j.$$

On the other hand, by the Bianchi identity,

$$\begin{aligned}
\mathrm{d}\Omega_{ij} &= \mathrm{d}\left(\Omega_i^l g_{lj}\right) \\
&= \mathrm{d}\Omega_i^l \cdot g_{lj} + \Omega_i^l \wedge \mathrm{d}g_{lj} \\
&= \left(\omega_i^k \wedge \Omega_k^l - \Omega_i^k \wedge \omega_k^l\right) g_{lj} + \Omega_i^l \wedge (\omega_{lj} + \omega_{jl}) \\
&= \omega_i^k \wedge \Omega_{kj} + \Omega_i^k \wedge \omega_{jk} \\
&= \omega_i^k \wedge \Omega_{kj} + \Omega_{ik} \wedge \omega_j^k.
\end{aligned}$$

Thus

$$\mathrm{d}\Omega_{ij} = -K\omega_i^k \wedge \theta_k \wedge \theta_j - K\theta_i \wedge \theta_k \wedge \omega_j^k = -K\mathrm{d}\theta_i \wedge \theta_j + K\theta_i \wedge \mathrm{d}\theta_j.$$

We then obtain

$$\mathrm{d}K \wedge \theta_i \wedge \theta_j = 0.$$

Since $\{\theta_i\}$ and $\{\mathrm{d}u^i\}$ are both local coframes, we may assume that $\mathrm{d}K = a^k\theta_k$. Since $m \geq 3$, we have

$$a^k\theta_1 \wedge \cdots \wedge \theta_m = (-1)^{k-1}\mathrm{d}K \wedge \theta_1 \wedge \cdots \wedge \widehat{\theta_k} \wedge \cdots \wedge \theta_m = 0, \quad 1 \leq k \leq m.$$

Hence $\mathrm{d}K = 0$. Since M is a connected manifold, K is a constant function on M. □

4.4 The Gauss-Bonnet Theorem

Suppose M is an oriented 2-dimensional Riemannian manifold. If we choose a smooth frame field $\{e_1, e_2\}$ in a coordinate neighborhood U whose orientation is consistent with that of M, with coframe $\{\theta^1, \theta^2\}$, then the Riemannian metric is

$$\mathrm{d}s^2 = g_{ij}\theta^i\theta^j, \quad 1 \leq i, j \leq 2,$$

where $g_{ij} = G(e_i, e_j)$. By the Fundamental Theorem of Riemannian Geometry, there exists a unique set of differential 1-forms θ_i^j such that

$$\mathrm{d}\theta^i - \theta^j \wedge \theta_j^i = 0, \quad \mathrm{d}g_{ij} = g_{ik}\theta_j^k + g_{kj}\theta_i^k.$$

The θ_i^j define the Levi-Civita connection on M by

$$\mathrm{D}e_i = \theta_i^j e_j.$$

The curvature form for the connection is

$$\Omega_i^j = \mathrm{d}\theta_i^j - \theta_i^k \wedge \theta_k^j.$$

Let $\Omega_{ij} = \Omega_i^k g_{kj}$, then Ω_{ij} is skew-symmetric. Since the indices i, j only take the values 1 and 2, the only nonzero element in the curvature form Ω_{ij} is Ω_{12}.

Let Ω denote the curvatre matrix (Ω_i^j) and write

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}.$$

If (e_1', e_2') is another local frame field in a coordinate neighborhood $W \subset M$ with orientation consistent with that of M, then in $U \cap W$, when $U \cap W \neq \varnothing$,

$$\begin{pmatrix} e_1' \\ e_2' \end{pmatrix} = A \cdot \begin{pmatrix} e_1 \\ e_2 \end{pmatrix},$$

where

$$A = \begin{pmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{pmatrix}, \quad \det A > 0.$$

Let G' and Ω' denote the corresponding quantities with respect to the frame field (e_1', e_2'). Then

$$G' = A \cdot G \cdot A^T, \quad \Omega' = A \cdot G \cdot A^{-1}.$$

Therefore

$$\Omega' \cdot G' = A \cdot (\Omega \cdot G) \cdot A^T,$$

i.e.

$$\begin{pmatrix} 0 & \Omega_{12}' \\ -\Omega_{12}' & 0 \end{pmatrix} = \begin{pmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{pmatrix} \begin{pmatrix} 0 & \Omega_{12} \\ -\Omega_{12} & 0 \end{pmatrix} \begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix}.$$

Thus

$$\Omega_{12}' = (a_1^1 a_2^2 - a_1^2 a_2^1)\Omega_{12} = (\det A) \cdot \Omega_{12}.$$

We also have

$$g' = \det G' = (\det A)^2 \cdot \det G = (\det A)^2 \cdot g.$$

Hence

$$\frac{\Omega_{12}'}{\sqrt{g'}} = \frac{\Omega_{12}}{\sqrt{g}}.$$

In the other words, $\Omega_{12}/\sqrt{g}$ is independent of the choice of the orientation-consistent local frame field, and is therefore an exterior differential 2-form defined on the whole manifold. If we choose a local coordinate system u^i with the same orientation as M, and $\{e_1, e_2\}$ is the natural basis, then

$$\Omega_{12} = \frac{1}{2} R_{12kl} \mathrm{d}u^k \wedge \mathrm{d}u^l = R_{1212} \mathrm{d}u^1 \wedge \mathrm{d}u^2.$$

Thus

$$\frac{\Omega_{12}}{\sqrt{g}} = \frac{R_{1212}}{g} \cdot \sqrt{g}\mathrm{d}u^1 \wedge \mathrm{d}u^2 = -K\mathrm{d}\sigma,$$

where K is the Gauss curvature of M and $\mathrm{d}\sigma = \sqrt{g}\mathrm{d}u^1 \wedge \mathrm{d}u^2$ is the oriented area element of M.

If $\{e_1, e_2\}$ is an orthogonal local frame field with an orientation consistent with that of M, then

$$g = g_{11}g_{22} - g_{12}^2 = 1.$$

Thus

$$K\mathrm{d}\sigma = -\Omega_{12}.$$

On the other hand,

$$\Omega_{12} = \mathrm{d}\theta_{12} + \theta_1^i \wedge \theta_{2i}.$$

The skew-symmetry of θ_i^j implies that

$$\Omega_{12} = \mathrm{d}\theta_{12},$$

where $\theta_{12} = \mathrm{D}e_1 \cdot e_2$. It then follows that

$$K\mathrm{d}\sigma = -\mathrm{d}\theta_{12}.$$

As long as there exists a smooth orthogonal frame field $\{e_1, e_2\}$ with an orientation consistent with M in an open subset $U \subset M$, then there exists a connection form θ_{12} on U, and hence the above formula holds.

On an oriented 2-dimensional Riemannian manifold, a smooth orthogonal frame field with an orientation consistent with that of M corresponds to a tangent vector field that is never zero. In fact, the tangent vector e_2 in the frame $\{e_1, e_2\}$ is obtained by rotating e_1 by $\pi/2$ according to the orientation of M. Therefore an orthogonal frame field $\{e_1, e_2\}$ with an orientation consistent with that of M is equivalent to the unit tangent vector field e_1.

A null point of a tangent vector field is called a **singular point**. Assume that there is a smooth vector field X on U that has exactly one singular point p, i.e. $X_q \neq 0$ whenever $q \in U - \{p\}$. Then there is a smooth unit tangent vector field

$$a_1 = \frac{X}{|X|}$$

which determines an orthogonal frame field $\{e_1, e_2\}$ with an orientation consistent with that of M in $U - \{p\}$. Therefore, if $\{e_1, e_2\}$ is a given orthogonal frame field on U that is also orientation-consistent with M, then we may assume that

$$a_1 = e_1 \cos\alpha + e_2 \sin\alpha,$$
$$a_2 = -e_1 \sin\alpha + e_2 \cos\alpha,$$

where $\alpha = \angle(e_1, a_1)$ is the oriented angle from e_1 to a_1. Although α is a multi-valued function, the difference between two values of α is an integer multiple of 2π at every point. Thus there always exists a continuous branch of α in a neighborhood of any point. The single-valued function obtained from this branch is smooth in the neighborhood. Let

$$\omega_{12} = \mathrm{D}a_1 \cdot a_2,$$

then direct calculation yields that

$$\omega_{12} = \mathrm{d}\alpha + \theta_{12}.$$

Suppose D is a simply connected domain containing the point p whose boundary is a smooth simple closed curve $C = \partial D$. Then C has a induced orientation of M. Suppose the arc length parameter of C is $s, 0 \le s \le L$, and the direction along the curve as s increases is the same as the induced direction of C. So $C(0) = C(L)$. Since C is compact, it can be covered by finitely many neighborhoods, and there exists a continuous branch of α in each neighborhood. Therefore, there exists a continuous function

$$\alpha = \alpha(s), \quad 0 \le s \le L$$

on C. By the Fundamental Theorem of Calculus we have

$$\alpha(L) - \alpha(0) = \int_0^L \mathrm{d}\alpha.$$

Since $\alpha(L)$ and $\alpha(0)$ are the angles between the tangent vectors a_1 and e_1 at the same point $C(0) = C(L)$, the left hand side is an integer multiple of 2π, and is independent of the choice of the continuous branch of $\alpha(s)$. It is also independent of the choice of the frame field $\{e_1, e_2\}$.

The value of

$$\alpha(L) - \alpha(0) = \int_0^L \mathrm{d}\alpha$$

given above is also independent of the choice of the simple closed curve C surrounding the point p. Suppose there is another simply connected domain

$D_1 \subset \mathring{D}$ containing p. Let $C_1 = \partial D_1$. Then $D - D_1$ is a domain with boundary in M, and its boundary with induced orientation is $C - C_1$. By the Stokes' Formula, we have

$$\begin{aligned}\int_{C-C_1} \mathrm{d}\alpha &= \int_{C-C_1} \omega_{12} - \int_{C-C_1} \theta_{12} \\ &= \int_{C-C_1} \omega_{12} - \int_{D-D_1} \mathrm{d}\theta_{12} \\ &= \int_{C-C_1} \omega_{12} + \int_{D-D_1} K \mathrm{d}\sigma.\end{aligned}$$

The right hand side is independent of the choice of the frame field $\{e_1, e_2\}$ on $D - D_1$. Hence we may assume that $e_i = a_i, i = 1, 2$. Then the right hand side vanishes and hence

$$\int_{C-C_1} \mathrm{d}\alpha = 0,$$

or equivalently,

$$\int_C \mathrm{d}\alpha = \int_{C_1} \mathrm{d}\alpha.$$

Definition 4.4.1 Suppose X is a smooth tangent vector field with an isolated singular point p, and U is a coordinate neighborhood of p such that p is the only singular point of X in U. Then the integer

$$I_p = \frac{1}{2\pi}[\alpha(L) - \alpha(0)] = \frac{1}{2\pi} \int_C \mathrm{d}\alpha,$$

obtained by the above construction is independent of the choice of the simple closed curve C surrounding p, and the choice of the frame field $\{e_1, e_2\}$ on U. It is called the **index** of the tangent vector field X at the point p.

Integrating

$$\omega_{12} = \mathrm{d}\alpha + \theta_{12}$$

over C we obtain

$$\frac{1}{2\pi} \int_C \omega_{12} = \frac{1}{2\pi} \int_C \mathrm{d}\alpha - \frac{1}{2\pi} \int_D K \mathrm{d}\sigma.$$

Since the Gauss curvature is continuous at p, when D is shrunk to a point, the integral

$$\frac{1}{2\pi} \int_D K \mathrm{d}\sigma \to 0.$$

However, the integral

$$\frac{1}{2\pi}\int_C \mathrm{d}\alpha$$

is exactly the constant I_p. Hence we have

$$I_p = \frac{1}{2\pi}\lim_{C\to p}\int_C \omega_{12}.$$

Theorem 4.4.1 (Gauss-Bonnet Theorem) Suppose M is a compact oriented 2-dimensional Riemannian manifold. Then

$$\frac{1}{2\pi}\int_M K\mathrm{d}\sigma = \chi(M),$$

where $\chi(M)$ is the **Euler characteristic** of M.

Proof. Choose a smooth tangent vector field X on M with only finitely many isolated singular points $p_i, 1 \leq i \leq r$. For each p_i, we choose a ε-ball neighborhood D_i, where ε is a sufficiently small positive number such that p_i is the only singular point of X in D_i. Let $C_i = \partial D_i$, then C_i is a simple closed curve with induced orientation from M on D_i. Thus the tangent vector field X determines a smooth orthogonal frame field $\{e_1, e_2\}$ on $M - \bigcup_i D_i$ that is orientation consistent, with $e_1 = X/|X|$. Suppose $\theta_{12} = \mathrm{D}e_1 \cdot e_2$. On $M - \bigcup_i D_i$, we have

$$\mathrm{d}\theta_{12} = \Omega_{12} = -K\mathrm{d}\sigma.$$

Also, by the Stokes' Formula,

$$\int_{M-\bigcup_i D_i} K\mathrm{d}\sigma = -\int_{M-\bigcup_i D_i} \mathrm{d}\theta_{12} = \sum_{i=1}^{r}\int_{C_i}\theta_{12}.$$

Since the frame field $\{e_1, e_2\}$ is actually well-defined on $M - \{p_i, 1 \leq i \leq r\}$, the equation still holds as $\varepsilon \to 0$. Also, since K is a continuously differentiable function defined on the whole M, we have

$$\lim_{\varepsilon\to 0}\int_{M-\bigcup_i D_i} K\mathrm{d}\sigma = \int_M K\mathrm{d}\sigma.$$

Noting that we also have

$$\lim_{\varepsilon\to 0}\sum_{i=1}^{r}\int_{C_i}\theta_{12} = 2\pi\sum_{i=1}^{r} I_{p_i},$$

it follows that

$$\frac{1}{2\pi}\int_M K \mathrm{d}\sigma = \sum_{i=1}^{r} I_{p_i}.$$

Since the left hand side is independent of the tangent vector field X, we may construct a special one as follows. Choose a triangulation of M with f faces, e edges and v vertices. Then we can construct a smooth tangent vector field X such that the center of mass of each face, the midpoint of each edge, and each vertex is a singular point, whose index is +1, -1, and +1, respectively. For this tangent vector we have

$$\sum_{i=1}^{r} I_{p_i} = f - e + v = \chi(M).$$

Hence

$$\frac{1}{2\pi}\int_M K \mathrm{d}\sigma = \chi(M).$$

□

The above proof also implies the **Hopf's Index Theorem** below.

Theorem 4.4.2 (Hopf's Index Theorem) Suppose there is a smooth tangent vector field on a compact oriented 2-dimensional Riemannian manifold with finitely many singular points. Then the sum of its indices at the various singular points is equal to the Euler characteristic of the manifold.

Suppose C is a smooth curve on M, and a_1 is a unit tangent vector to C. Choose a unit normal vector a_2 to C such that the orientation determined by $\{a_1, a_2\}$ is consistent with that of M. Since $\mathrm{D}a_1$ is colinear with a_2, we may assume

$$\kappa_g = \frac{\mathrm{D}a_1}{\mathrm{d}s} \cdot a_2.$$

κ_g is called the **geodesic curvature** of C. A necessary and sufficient condition for C to be a geodesic curve is

$$\kappa_g \equiv 0.$$

Suppose D is a compact domain with boundary in an oriented 2-dimensional Riemannian manifold M whose boundary ∂D is composed of finitely many piecewise smooth simple closed curves with induced orientation from D.

Suppose the interior angle of ∂D at each vertex p_i is $\alpha_i, 1 \leq i \leq l$. By the similar method we can prove the **Gauss-Bonnet Formula**

$$\sum_{i=1}^{l}(\pi - \alpha_i) - \int_{\partial D} \kappa_g \mathrm{d}s + \int_D K \mathrm{d}\sigma = 2\pi \cdot \chi(D),$$

where κ_g is the geodesic curvature along ∂D. If D is a geodesic triangle in M, and ∂D is a closed curve composed of three geodesic segments, then $\chi(D) = 1$ and therefore

$$\alpha_1 + \alpha_2 + \alpha_3 - \pi = \int_D K \mathrm{d}\sigma.$$

5 Lie Groups

5.1 Lie Groups

Definition 5.1.1 Let G be a nonempty set. If

1. G is a group;
2. G is an r-dimensional smooth manifold; and
3. the inverse map $\tau : G \to G$ such that $\tau(g) = g^{-1}$ and the multiplication map $\varphi : G \times G \to G$ such that $\varphi(g_1, g_2) = g_1 \cdot g_2$ are both smooth maps,

then G is called an r-dimensional **Lie group**.

Since $\tau^2 = \mathrm{id} : G \to G$, τ is a diffeomorphism from G to itself. For $ginG$, the **right translation** by g on G is $R_g : G \to G$ such that $R_g(x) = \varphi(x, g) = x \cdot g$, and the **left translation** is $L_g : G \to G$ such that

$$L_g(x) = \varphi(g, x) = g \cdot x.$$

Since the inverse of L_g is $L_{g^{-1}}$ and the inverse of R_g is $R_{g^{-1}}$, L_g and R_g are both diffeomorphisms from G to itself.

If G_1, G_2 are Lie groups, then the product manifold $G_1 \times G_2$ can also be viewed as the product of groups. Therefore $G_1 \times G_2$ is also a Lie group, called the **direct product** of the Lie groups G_1 and G_2.

Example 5.1.1 $\mathrm{GL}(n;\mathbb{R})$ is the set of nondegenerate $n \times n$ real matrices with matrix multiplication for its group operation. Since $\mathrm{GL}(n;\mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$, it has the differentiable structure induced from $\mathbb{R}^{n^2}$. Suppose

$$A = (A_i^j), \quad B = (B_i^j) \in \mathrm{GL}(n;\mathbb{R}).$$

Then

$$(A \cdot B)_i^j = A_i^k B_k^j.$$

Since the right hand side is a polynomial of the elements of the matrices A and B, the map

$$\varphi(A, B) = A \cdot B$$

is smooth. Moreover, since the elements of A^{-1} are rational functions of the elements A_i^j, the inverse map is also smooth. Hence $\mathrm{GL}(n;\mathbb{R})$ is an n^2-dimensional Lie group, called the **general linear group**. Similarly the multiplicative group $\mathrm{GL}(n;\mathbb{C})$ of nondegenerate $n \times n$ complex matrices is a $2n^2$-dimensional Lie group.

Example 5.1.2 Suppose G is a Lie group and H is a subgroup of G. If H is regular submanifold of G, then it can be shown that the restrictions of the multiplicaiton map and the inverse map, namely

$$\varphi|_{H\times H} : H \times H \to H, \quad \tau|_H : H \to H,$$

are both smooth.

Suppose

$$\mathrm{SL}(n;\mathbb{R}) = \{A \in \mathrm{GL}(n;\mathbb{R}) \mid \det A = 1\}$$

and

$$\mathrm{O}(n;\mathbb{R}) = \{A \in \mathrm{GL}(n;\mathbb{R}) \mid A \cdot A^T = I\}.$$

Then $\mathrm{SL}(n;\mathbb{R})$ and $\mathrm{O}(n;\mathbb{R})$ are both subgroups and regular submanifolds of $\mathrm{GL}(n;\mathbb{R})$. Therefore they are Lie groups. $\mathrm{SL}(n;\mathbb{R})$ and $\mathrm{O}(n;\mathbb{R})$ are called the **special linear group** and the **real orthogonal group**, respectively.

Suppose G is an r-dimensional Lie group with identity e. Since for every $a \in G$, the map $R_{a^{-1}}$ is a diffeomorphism from G to itself that takes a to e, the tangent map $(R_{a^{-1}})_* : G_a \to G_e$ is a linear isomorphism, where G_a is the tangent space of G at a. Suppose $X \in G_a$. Let

$$\omega(X) = (R_{a^{-1}})_* X.$$

Then ω is a differential 1-form defined on G with values in G_e, called the **right fundamental differential form** or **Maurer-Cartan form** of the Lie group G. If we choose a basis $\delta_i, 1 \le i \le r$ for G_e, then we may write

$$\omega = \omega^i \delta_i,$$

where $\omega^i, 1 \le i \le r$ are r differential 1-forms on G that are linearly independent everywhere.

Choose a local coordinate system $(U; x^i)$ and $(W; y^i)$ at points e and a, respectively. When U is sufficiently small, there exists a neighborhood $W_1 \subset W$ of a such that $\varphi(U \times W_1) \subset W$. Choose

$$\delta_i = \left.\frac{\partial}{\partial x^i}\right|_e$$

and let

$$\varphi^i(x, y) = y^i \circ \varphi(x, y), \quad (x, y) \in U \times W_1.$$

Then the isomorphism $(R_a)_* : G_e \to G_a$ is given as

$$(R_a)_* \delta_i = \left.\frac{\partial \varphi^j(x, a)}{\partial x^i}\right|_{x=e} \cdot \left.\frac{\partial}{\partial y^j}\right|_a .$$

Because

$$(R_{a^{-1}})_* \circ (R_a)_* = \mathrm{id} : G_e \to G_e,$$

we have

$$(R_{a^{-1}})_* \left.\frac{\partial}{\partial y^i}\right|_a = \Lambda_i^j(a) \delta_j,$$

Where $(\Lambda_i^j(a))$ is the inverse matrix of $((\partial \varphi^i(x, a)/\partial y^j)_{x=e})$. Therefore

$$\omega^i = \Lambda_j^i(a) \cdot \mathrm{d}y^j,$$

hence ω^i is a smooth differential 1-form.

Theorem 5.1.1 Suppose $\sigma : G \to G$ is a smooth map. If σ is a right translation of the Lie group G, then it preserves the right fundamental differential form, i.e.,

$$\sigma^* \omega^i = \omega^i, \quad 1 \le i \le r.$$

Proof. Suppose σ is the right translation R_x for some $x \in G$. Then for any $X \in G_a$ we have

$$\begin{aligned} ((R_x)^* \omega)(X) &= \omega((R_x)_* X) \\ &= (R_{(ax)^{-1}})_* \circ (R_x)_* X \\ &= (R_{a^{-1}})_* X \\ &= \omega(X). \end{aligned}$$

Hence

$$(R_x)_* \omega = \omega.$$

□

Because $\mathrm{d}\circ\sigma^* = \sigma^*\circ\mathrm{d}$ holds for any smooth map $\sigma : G \to G$, $\mathrm{d}\omega^i$ is still invariant under right-translation. Let

$$\mathrm{d}\omega^i = -\frac{1}{2}c^i_{jk}\omega^j \wedge \omega^k,$$

where

$$c^i_{jk} + c^i_{kj} = 0.$$

Because ω^i and $\mathrm{d}\omega^i$ are both right-invariant, the c^i_{jk} are constants, called the **structure constants** of the Lie group. The above equation is called the **structure equation** or the **Maurer-Cartan equation** of the Lie group G.

Theorem 5.1.2 The structure constants c^i_{jk} satisfy the Jacobi identity

$$c^i_{jk}c^j_{hl} + c^i_{jh}c^j_{lk} + c^i_{jl}c^j_{kh} = 0.$$

Proof. Exteriorly differentiating

$$\mathrm{d}\omega^i = -\frac{1}{2}c^i_{jk}\omega^j \wedge \omega^k,$$

we get

$$\begin{aligned} 0 &= -\frac{1}{2}c^i_{jk}(\mathrm{d}\omega^j \wedge \omega^k - \omega^j \wedge \mathrm{d}\omega^k) \\ &= \frac{1}{2}c^i_{jk}c^j_{hl}\omega^h \wedge \omega^l \wedge \omega^k \\ &= \frac{1}{6}(c^i_{jk}c^j_{hl} + c^i_{jh}c^j_{lk} + c^i_{jl}c^j_{kh})\omega^h \wedge \omega^l \wedge \omega^k. \end{aligned}$$

The terms inside the parentheses are skew-symmetric with respect to k, h, l. Hence the Jacobi identity follows. □

Definition 5.1.2 Suppose X is a smooth tangent vector field on a Lie group G. If, for any $a \in G$, we have

$$(R_a)_*X = X,$$

then we say that the tangent vector field X is a **right-invariant vector field** on G.

Choose an arbitrary tangent vector $X_e \in G_e$, and let $X_a = (R_a)_* X_e$ for each $a \in G$. Then we obtain a smooth tangent vector field X on G. For any $a, x \in G$, we have

$$(R_x)_* X_a = (R_x)_* \circ (R_a)_* X_e = (R_{ax})_* X_e = X_{ax},$$

hence X is right-invariant. Let X_i denote the right-invariant vector field obtained by the right translation of $\delta_i \in G_e$. Then the $X_i, 1 \leq i \leq r$ are tangent vector fields which are linearly independent everywhere on G, and any right-invariant vector field on G can be expressed as a linear combination of the X_i with constant coefficients. Hence the set of right-invariant vector fields on G forms an r-dimensional vector space, denoted by $\mathcal{G}$, and is isomorphic to G_e.

By the construction of X_i we have

$$\omega(X_i) = \delta_i,$$

that is,

$$\omega^j(X_i) = \langle X_i, \omega^j \rangle = \delta_i^j.$$

Thus the fundamental differential forms $\omega^i, 1 \leq i \leq r$ and the right-invariant vector fields $X_j, 1 \leq j \leq r$ constitute sets of mutually dual coframe fields and frame fields, respectively, on the Lie group G. Therefore a tangent vector field X on G is right-invariant if and only if the value of the right fundamental form on X is constant.

Theorem 5.1.3 If X, Y are right-invariant vector fields on G, then $[X, Y]$ is also a right-invariant vector field on G.

Proof. First we have

$$\langle X \wedge Y, \mathrm{d}\omega^i \rangle = X\langle Y, \omega^i \rangle - Y\langle X, \omega^i \rangle - \langle [X, Y], \omega^i \rangle$$

from Lemma 3.1.3. From the structure equation we obtain

$$\langle X \wedge Y, \mathrm{d}\omega^i \rangle = -\frac{1}{2} c^i_{jk} \langle X \wedge Y, \omega^j \wedge \omega^k \rangle = -c^i_{jk} \omega^j(X) \omega^k(Y).$$

Since X, Y are both right-invariant vector fields, we have $\omega^j(X), \omega^k(Y)$ are both constant. Therefore

$$\omega^i([X, Y]) = c^i_{jk} \omega^j(X) \omega^k(Y)$$

is also constant. This implies that $[X, Y]$ is right-invariant. □

The Poisson bracket is then closed in $\mathcal{G}$ and defines a multiplication operation on $\mathcal{G}$, which satisfies the following conditions:

1. Distributive Law: $[a_1X_1 + a_2X_2, Y] = a_1[X_1, Y] + a_2[X_2, Y]$;
2. Skew-symmetric Law: $[X, Y] = -[Y, X]$;
3. Jacobi Identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

If an n-dimensional real vector space has a multiplication operation satisfying the distributive law, the skew-symmetric law and the Jacobi identity, then we call it an n-dimensional **Lie algebra**. Then vector space $\mathcal{G}$ of all right-invariant vector fields on a Lie group G is an r-dimensional Lie algebra, called the **Lie algebra** of the Lie group G.

The structure constants of a Lie group provide the multiplication table for its Lie algebra $\mathcal{G}$. In fact, by the proof of the Theorem 5.1.3, we have

$$\omega^i([X_j, X_k]) = c^i_{jk},$$

and then

$$[X_j, X_k] = c^i_{jk} X_i.$$

The skew-symmetry of the structure constants c^i_{jk} with respect to the lower indices and the Jacobi identity satisfied by these constants correspond to the skew-symmetry of the Poisson bracket and its Jacobi identity. Thus if we let

$$[\delta_j, \delta_k] = c^i_{jk} \delta_i,$$

then G_e also becomes an r-dimensional Lie algebra, and G_e and $\mathcal{G}$ are isomorphic as Lie algebras. Usually the Lie algebra G_e is also called the **Lie algebra** of the Lie group G.

Example 5.1.3 Suppose $A = (A^j_i) \in \mathrm{GL}(n; \mathbb{R})$. Then $A^j_i, 1 \le i, j \le n$ is a coordinate system on the manifold $\mathrm{GL}(n; \mathbb{R})$, and then $\mathrm{d}A^j_i, 1 \le i, j \le n$ gives a coframe field on $\mathrm{GL}(n; \mathbb{R})$. The right fundamental differential form of $\mathrm{GL}(n; \mathbb{R})$ can be written as

$$\omega = \mathrm{d}A \cdot A^{-1}.$$

Exterior differentiation then yields

$$\begin{aligned} \mathrm{d}\omega = -\mathrm{d}A \wedge \mathrm{d}A^{-1} &= -(\mathrm{d}A \cdot A^{-1}) \wedge (A \cdot \mathrm{d}A^{-1}) \\ &= (\mathrm{d}A \cdot A^{-1}) \wedge (\mathrm{d}A \cdot A^{-1}) = \omega \wedge \omega. \end{aligned}$$

Let $\mathrm{gl}(n;\mathbb{R})$ denote the tangent space at the identity element I in the Lie group $\mathrm{GL}(n;\mathbb{R})$. It is the n^2-dimensional vector space with $n \times n$ real matrices as its elements. In this representation, $\mathrm{gl}(n;\mathbb{R})$ has a basis $E_i^j, 1 \leq i, j \leq n$, where E_i^j denote the $n \times n$ matrix with the value 1 for the element at the intersection of the j-th row and the i-th column, and 0 for other entries. Hence we may write

$$\omega = \omega_i^j E_j^i = (\omega_i^j).$$

From

$$\mathrm{d}\omega = \omega \wedge \omega$$

we have

$$\mathrm{d}\omega_i^j = \omega_i^k \wedge \omega_k^j = \frac{1}{2}(\delta_i^p \delta_q^j \delta_s^r - \delta_i^r \delta_s^j \delta_q^p)\omega_p^s \wedge \omega_r^q.$$

Hence the structure constants of the Lie group $\mathrm{GL}(n;\mathbb{R})$ are

$$c_{(p,s)(r,q)}^{(i,j)} = -\delta_i^p \delta_q^j \delta_s^r + \delta_i^r \delta_s^j \delta_q^p.$$

The multiplication table for the Lie algebra $\mathrm{gl}(n;\mathbb{R})$ is then

$$[E_s^p, E_q^r] = \delta_q^p E_s^r - \delta_s^r E_q^p = E_q^r \cdot E_s^p - E_s^p \cdot E_q^r.$$

Suppose $A, B \in \mathrm{gl}(n;\mathbb{R})$, then the above formula implies that

$$[A, B] = B \cdot A - A \cdot B.$$

Definition 5.1.3 Suppose G, H are two Lie groups. If there is a smooth map $f : H \to G$ which is also a homomorphism between the groups, then f is called a **homomorphism** of Lie groups from H to G. If f is also a diffeomorphism, then it is called an **isomorphism** of Lie groups from H to G.

Theorem 5.1.4 Suppose $f : H \to G$ is a Lie group homomorphism, then f induces a homomorphism $f_* : \mathcal{H} \to \mathcal{G}$ between the Lie algebras. If f is a Lie group isomorphism, then f_* is an isomorphism betweeen the Lie algebras.

Proof. Let f_* denote the tangent map of the smooth map f. First we show that f_* maps the right-invariant vector fields of the Lie group H to the right-invariant vector fields of the Lie group G. Choose any $X_e \in H_e$, and let $Y_{e'} = f_* X_e \in G_{e'}$, where e is the identity element of H and $e' = f(e)$ is the identity element of G. Let X, Y be the right-invariant vector fields generated by $X_e, Y_{e'}$ on their respective Lie groups. Then for any $a \in H$, we have

$$f_* X_a = f_* \circ (R_a)_* X_e = (R_{a'})_* \circ f_* X_e = (R_{a'})_* Y_{e'} = Y_{a'},$$

where $a' = f(a) \in G$. Thus the image of a right-invariant vector fields on H under f_* can be extended to a right-invariant vector field on G. Use the notation $f_* : \mathcal{H} \to \mathcal{G}$ for this correspondence. Since the tangent map f_* commutes with the Poisson bracket product of vector fields. Hence $f_* : \mathcal{H} \to \mathcal{G}$ defined above is a homomorphism between Lie algebras.

When f is an isomorphism between Lie groups, f_* is also invertible and hence is an isomorphism between Lie algebras. □

Suppose G is an r-dimensional Lie group. A homomorphism from the Lie group G to $\mathrm{GL}(n;\mathbb{R})$ is called a **representation** of order n of the Lie group G. A natural representation of order r for each r-dimensional Lie group can be defined as follows.

Suppose $x \in G$, and let

$$\alpha_x(g) = x \cdot g \cdot x^{-1} = L_x \circ R_{x^{-1}}(g).$$

Then α_x is an automorphism of the Lie group G, called the **inner automorphism** of G. The tangent map $(\alpha_x)_*$ of α_x determines an automorphism of the Lie algebra G_e. Let $\mathrm{Ad}(x) = (\alpha_x)_* : G_e \to G_e$, then $\mathrm{Ad}(x)$ is a nondegenerate linear transformation on the linear space G_e, and is therefore an element of $\mathrm{GL}(r;\mathbb{R})$. Hence we obtain a map $\mathrm{Ad} : G \to \mathrm{GL}(r;\mathbb{R})$. It can be verified that Ad is a homomorphism between groups. If we use local coordinates, Ad is given by smooth functions of the local coordinates, hence Ad is a homomorphism between Lie groups.

Definition 5.1.4 The Lie group homomorphism $\mathrm{Ad} : G \to \mathrm{GL}(r;\mathbb{R})$ given above is called the **adjoint representation** of the Lie group G.

The tangent map of the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(r;\mathbb{R})$ induces a homomorphism ad from the Lie algebra G_e to $\mathrm{gl}(r;\mathbb{R})$, called the **adjoint representation** of the Lie algebra G_e of the Lie group G. Since $\mathrm{gl}(r;\mathbb{R})$ can be viewed as a set of linear transformations on G_e, the adjoint representation ad actually assigns to each $X \in G_e$ a linear transformation $\mathrm{ad}(X)$ on G_e.

References

[1] S. S. Chern, W. H. Chen, and K. S. Lam. *Lectures on Differential Geometry*. World Scientific, 2000.