Sheaf Theory

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We are trying to use the categorical language to describe the sheaf theory, mostly using the universal properties.

1 Sheaves of sets

Construct a contravariant functor $\mathsf{Open} : \mathsf{Top}^{\mathrm{op}} \to \mathsf{Cat}$. For each topological space X, we define $\mathsf{Open}(X)$ to be the category whose objects are open subsets of X and whose morphisms are inclusions of open sets. For each continuous map $f : X \to Y$, the functor $\mathsf{Open}(f)$ sends an open set $V \subset Y$ to the open set $f^{-1}(V) \subset X$, and maps the inclusions correspondingly.

Definition 1.1 (presheaf) A **presheaf (of sets)** on a topological space X is a contravariant functor

$$\mathcal{F} : \mathsf{Open}(X)^{\mathrm{op}} \to \mathsf{Set}.$$

The category of presheaves on X, with morphisms given by natural transformations, is denoted by $\mathsf{pSh}(X, \mathsf{Set})$, i.e.,

$$\mathsf{pSh}(X, \mathsf{Set}) = \mathsf{Fun}(\mathsf{Open}(X)^{\mathrm{op}}, \mathsf{Set}).$$

If $\mathcal{F}$ is a presheaf on X and $U \subset V \subset X$ are open sets, then we denote the **restriction morphism**

$$\mathcal{F}(V) \to \mathcal{F}(U)$$

by r_U^V or $(-)|_U$.

Definition 1.2 (sheaf) A **sheaf (of sets)** on a topological space X is a presheaf $\mathcal{F}$ such that for every open cover $\{U_i\}_{i\in I}$ of an open set $U \subset X$, the following diagram is an equalizer:

$$\mathcal{F}(U) \xrightarrow{\pi} \prod_{i\in I} \mathcal{F}(U_i) \underset{\rho}{\overset{\lambda}{\rightrightarrows}} \prod_{j,k\in I} \mathcal{F}(U_j \cap U_k)$$

where π is given by the restriction morphisms $\mathcal{F}(U) \to \mathcal{F}(U_i)$, λ is given by the restriction morphisms $\mathcal{F}(U_i) \to \mathcal{F}(U_i \cap U_k)$ and ρ is given by the restriction morphisms $\mathcal{F}(U_i) \to \mathcal{F}(U_j \cap U_i)$.

The category of sheaves on X, with morphisms given by natural transformations, is denoted by $\mathsf{Sh}(X, \mathsf{Set})$. It is clear that $\mathsf{Sh}(X, \mathsf{Set})$ is a full subcategory of $\mathsf{pSh}(X, \mathsf{Set})$.

Example 1.1 (sheaf of sections) Suppose E and X are a topological spaces and $p : E \to X$ is a continuous map. For each open subset $U \subset X$, a **section** of (E, p) on U is a continuous map $s : U \to E$ such that $p(s(x)) = x$ for each $x \in U$. Denote the set of sections of (E, p) on U by $\Gamma(U, E)$. The assignment

$$U \mapsto \Gamma(U, E)$$

defines a presheaf on X, with the restriction morphisms given by the restriction of sections. This is actually a sheaf, called the **sheaf of sections** of (E, p).

The **sheafification** of presheaves is defined as the left adjoint of the inclusion (or forgetful) functor

$$\mathsf{Sh}(X, \mathsf{Set}) \to \mathsf{pSh}(X, \mathsf{Set}).$$

The sheafification of a presheaf $\mathcal{F}$ is usually denoted by $\mathcal{F}^+$.

The sheafification of a presheaf can be constructed explictly using the concept of etale spaces.

Definition 1.3 (etale space) An **etale space** over a topological space X is a pair (E, p), where E is a topological space and $p : E \to X$ is a local homeomorphism.

The category of etale spaces over X is denoted by $\mathsf{Et}(X)$. The morphisms in $\mathsf{Et}(X)$ are continuous maps $f : (E, p) \to (E', p')$ such that $p' \circ f = p$.

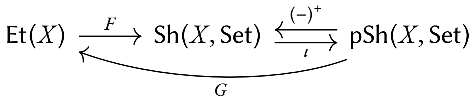
Theorem 1.1 For a topological space X, there are functors

$$F : \mathsf{Et}(X) \to \mathsf{Sh}(X, \mathsf{Set})$$

and

$$G : \mathsf{pSh}(X, \mathsf{Set}) \to \mathsf{Et}(X)$$

such that F is a category equivalence and the following diagrams commutes up to natural isomorphism:



where ι is the inclusion functor and $(-)^+$ is the sheafification functor.

Proof. The functor F is defined such that $F(E, p)$ is the sheaf of sections of (E, p), and the morphism $F(f)$ is given by the compostion of f with the sections.

Suppose $\mathcal{F}$ is a presheaf on X. For each $x \in X$, define the **stalk** of $\mathcal{F}$ at x to be the set

$$\mathcal{F}_x = \varinjlim \mathcal{F}(U),$$

where the limit is taken over all open sets U containing x. Let

$$E = \coprod_{x \in X} \mathcal{F}_x$$

and define the map

$$p : E \to X$$

by assigning $x \in X$ to the elements in the stalk $\mathcal{F}_x$. For each open set $U \subset X$ and $s \in \mathcal{F}(U)$, there is a function $\tilde{s} : U \to E$ mapping each $x \in U$ to the corresponding element of s in $\mathcal{F}_x$. Give E the finest topology such that $\tilde{s}$ is continuous for each open subset $U \subset X$ and $s \in \mathcal{F}(U)$. It can be verified that (E, p) is an etale space over X.

Now suppose $f : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves. Using the universal property of the inductive limit, we can deduce a unique map

$$\tilde{f}_x : \mathcal{F}_x \to \mathcal{G}_x$$

for each $x \in X$ such that for each neighborhood U of x, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f(U)} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_x & \xrightarrow{\tilde{f}_x} & \mathcal{G}_x \end{array}$$

Suppose (E, p) is the etale space constructed from $\mathcal{F}$ and (E', p') is the etale space constructed from $\mathcal{G}$. Putting these $\tilde{f}_x$ together, we get a morphism

$$\tilde{f} : (E, p) \to (E', p')$$

in $\mathsf{Et}(X)$. The functor G is then defined by assigning to each presheaf $\mathcal{F}$ the etale space (E, p) constructed above. □

For each presheave $\mathcal{F}$, the corresponding sheaf $F(G(\mathcal{F}))$ is a sheafification of $\mathcal{F}$, called the sheaf generated by the presheaf $\mathcal{F}$.

Remark Suppose (E, p) is an etale space over X and $\mathcal{F}$ is the sheaf of sections of (E, p). Then the elements in $\mathcal{F}_x$ is called the **germs** of sections of (E, p) at x. Two sections s, t of (E, p) on neighborhoods U and V of x define the same germ at x if and only if there exists a neighborhood $W \subset U \cap V$ of x such that $s|_W = t|_W$.

Remark The sheafification process preserves the stalks. Indeed, for each presheaf $\mathcal{F}$, the stalk of $\mathcal{F}^+$ at x is naturally isomorphic to

$$\varinjlim \Gamma\left(U, \coprod_{y \in X} \mathcal{F}_y\right) = \mathcal{F}_x,$$

where the limit is taken over all open sets U containing x. This isomorphism is actually canonical.

Using the category equivalence betwen $\mathsf{Et}(X)$ and $\mathsf{Sh}(X, \mathsf{Set})$, we can also figure out the condition of a morphism of sheaves to be injective ot surjective. Suppose $\mathcal{F}$ and $\mathcal{G}$ are sheaves on X and $f : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves. Then f is injective (surjective) if and only if the corresponding morphism of etale spaces $\tilde{f} : (E, p) \to (E', p')$ is injective (surjective), which is then equivalent to that $\tilde{f}_x$ is injective (surjective) for each $x \in X$. It can be seen that if f is injective, then $f(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for each open set $U \subset X$. However, if f is surjective, it is *not necessarily true* that $f(U)$ is surjective for each open set $U \subset X$.

The subsheaf and quotient sheaf of a sheaf $\mathcal{F}$ can be defined as the subobject and quotient object in the category $\mathsf{Sh}(X, \mathsf{Set})$. Specifically, a subsheaf of $\mathcal{F}$ is a sheaf $\mathcal{G}$ together with an injective morphism $i : \mathcal{G} \to \mathcal{F}$, and a quotient sheaf of $\mathcal{F}$ is a sheaf $\mathcal{Q}$ together with a surjective morphism $q : \mathcal{F} \to \mathcal{Q}$. Similarly we can consider the direct product of sheaves and the inductive limit of a family of sheaves.

Two important constructions in sheaf theory are the direct image sheaf and the inverse image sheaf.

Suppose $f : X \to Y$ is a continuous map between topological spaces X and Y. Then f induces a functor

$$\mathsf{Open}(f) : \mathsf{Open}(Y) \to \mathsf{Open}(X).$$

For each presheaf $\mathcal{F}$ on X, define the **direct image** $f_*\mathcal{F}$ to be the presheaf on Y given by the composition

$$\mathsf{Open}(Y) \xrightarrow{\mathsf{Open}(f)} \mathsf{Open}(X) \xrightarrow{\mathcal{F}} \mathsf{Set}\ .$$

It can be verified that $f_*\mathcal{F}$ is a sheaf on Y. The direct image f_* gives a functor

$$f_* : \mathsf{Sh}(X, \mathsf{Set}) \to \mathsf{Sh}(Y, \mathsf{Set}).$$

Suppose $\mathcal{F}$ is a sheaf on X and $\mathcal{G}$ is a sheaf on Y, with the corresponding etale spaces $p : E \to X$ and $p' : E' \to Y$. A morphism $g : \mathcal{F} \to \mathcal{G}$ compatible with f is a continuous map $g : E \to E'$ such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{g} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ X & \xrightarrow[f]{} & Y \end{array}$$

The **inverse image** $f^{-1}\mathcal{G}$ of $\mathcal{G}$ is the sheaf on X, together with a morphism $\bar{f} : f^{-1}\mathcal{G} \to \mathcal{G}$ compatible with f, satisfying the following universal property: for each sheaf $\mathcal{F}$ on X and each morphism $g : \mathcal{F} \to \mathcal{G}$ compatible with f, there exists a unique morphism $h : \mathcal{F} \to f^{-1}\mathcal{G}$ (of sheaves on X) such that $g = \bar{f} \circ h$. The inverse image f^{-1} gives a functor

$$f^{-1} : \mathsf{Sh}(Y, \mathsf{Set}) \to \mathsf{Sh}(X, \mathsf{Set}).$$

The inverse image $f^{-1}\mathcal{G}$ can be constructed explicitly by the map

$$U \mapsto f^{-1}\mathcal{G}(U) := \{s \in \Gamma(U, E') \mid s(x) \in \mathcal{G}_{f(x)} \text{ for each } x \in U\}.$$

An equivalent constuction of the inverse image is given by

$$f^{-1}\mathcal{G}(U) = \varinjlim \mathcal{G}(V),$$

where the inductive limit is taken over all open set $V \subset Y$ such that $f(U) \subset V$.

If X is a subspace of Y and $f : X \to Y$ is the inclusion map, then the inverse image $f^{-1}\mathcal{G}$ is called the **restriction sheaf** of $\mathcal{G}$ to X, and is denoted by $\mathcal{G}|_X$

Theorem 1.2 For each continuous map $f : X \to Y$ between topological spaces, the functors $f^{-1} : \mathsf{Sh}(Y, \mathsf{Set}) \to \mathsf{Sh}(X, \mathsf{Set})$ and $f_* : \mathsf{Sh}(X, \mathsf{Set}) \to \mathsf{Sh}(Y, \mathsf{Set})$ form an adjoint pair, i.e., there is a natural isomorphism for each sheaf $\mathcal{F}$ on X and each sheaf $\mathcal{G}$ on Y

$$\mathrm{Hom}_{\mathsf{Sh}(X,\mathsf{Set})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathsf{Sh}(Y,\mathsf{Set})}(\mathcal{G}, f_*\mathcal{F}).$$

2 Sheaves of modules

The construction of sheaves of sets can be generalized to sheaves of objects in an arbitrary category $\mathcal{C}$.

Definition 2.1 (presheaf) A **($\mathcal{C}$-valued) presheaf** on a topological space X is a contravariant functor

$$\mathcal{F} : \mathsf{Open}(X)^{\mathrm{op}} \to \mathcal{C}.$$

Definition 2.2 (sheaf) A **($\mathcal{C}$-valued) sheaf** on a topological space X is a presheaf $\mathcal{F}$ such that for every open cover $\{U_i\}_{i\in I}$ of an open set $U \subset X$, the following diagram is an equalizer:

$$\mathcal{F}(U) \xrightarrow{\pi} \prod_{i\in I} \mathcal{F}(U_i) \overset{\lambda}{\underset{\rho}{\rightrightarrows}} \prod_{j,k\in I} \mathcal{F}(U_j \cap U_k)$$

where π is given by the restriction morphisms $\mathcal{F}(U) \to \mathcal{F}(U_i)$, λ is given by the restriction morphisms $\mathcal{F}(U_i) \to \mathcal{F}(U_i \cap U_k)$ and ρ is given by the restriction morphisms $\mathcal{F}(U_i) \to \mathcal{F}(U_j \cap U_i)$.

The category of $\mathcal{C}$-valued presheaves and sheaves on X are also defined naturally, denoted by $\mathsf{pSh}(X, \mathcal{C})$ and $\mathsf{Sh}(X, \mathcal{C})$, respectively.

An Ab-valued (pre-) sheaf is also called a **sheaf of abelian groups**, and a Rng-valued (pre-) sheaf is called a **sheaf of rings**.

Similarly to the case of (pre-) sheaces of sets, we can define the stalk of a presheaf at a point, and the sheafification of a presheaf. The stalk of a sheaf $\mathcal{F}$ at $x \in X$ is denoted by $\mathcal{F}_x$, and the sheafification of $\mathcal{F}$ is denoted by $\mathcal{F}^+$.

Example 2.1 Suppose X is a topological space and A is an abelian group (or a ring). Consider the presheaf $\mathcal{F}$ on X given by $U \mapsto A$ for each nonempty open set $U \subset X$. Then the shefification $\mathcal{F}^+$ is a sheaf of abelian groups (or rings) on X, which has the expression as

$$\mathcal{F}^+(U) = \{f : U \to A \mid f \text{ is locally constant}\}.$$

This is called the **locally constant sheaf** on X with values in A, denoted by $\underline{A}$. The stalk of $\underline{A}$ at $x \in X$ is isomorphic to A for each $x \in X$.

Example 2.2 If X is a topological space, then the sheaf of continuous functions on X, given by

$$C^0_X(U) = \{f : U \to \mathbb{R} \mid f \text{ is continuous}\},$$

is a sheaf of rings on X. If M is a smooth manifold, then the sheaf of smooth functions on M, given by

$$C^\infty_M(U) = \{f : U \to \mathbb{R} \mid f \text{ is smooth}\},$$

is a sheaf of rings on M. If M is a complex manifold, then the sheaf of holomorphic functions on M, given by

$$\mathcal{O}_M(U) = \{f : U \to \mathbb{C} \mid f \text{ is holomorphic}\},$$

is a sheaf of rings on M.

Definition 2.3 (module over a shef of rings) Suppose X is a topological space and $\mathcal{A}$ is a sheaf of rings on X. A **(left) $\mathcal{A}$-module** is a sheaf $\mathcal{M}$ of abelian groups on X such that for each open set $U \subset X$, $\mathcal{M}(U)$ is a (left) $\mathcal{A}(U)$-module, and that for each open sets $V \subset U \subset X$, the following disgram commutes:

$$\begin{array}{ccc} \mathcal{A}(U) \times \mathcal{M}(U) & \longrightarrow & \mathcal{M}(U) \\ \downarrow & & \downarrow \\ \mathcal{A}(V) \times \mathcal{M}(V) & \longrightarrow & \mathcal{M}(V) \end{array}$$

where the vertical arrows are the restriction morphisms and the horizontal arrows are the action of $\mathcal{A}$ on $\mathcal{M}$.

Remark If $\mathcal{A} = \underline{A}$, then an $\underline{A}$-module is exactly a sheaf of A-modules on X. In particular, a $\underline{\mathbb{Z}}$-module is equivalent to a sheaf of abelian groups on X.

Remark By taking the inductive limits, we can see that for each $x \in X$, the stalk $\mathcal{M}_x$ is naturally a $\mathcal{A}_x$-module.

A morphism $f : \mathcal{L} \to \mathcal{M}$ of $\mathcal{A}$-modules is a morphism of sheaves of abelian groups such that for each open set $U \subset X$, $f(U) : \mathcal{L}(U) \to \mathcal{M}(U)$ is a homomorphism of $\mathcal{A}(U)$-modules. For two morphisms $f, g : \mathcal{L} \to \mathcal{M}$ of $\mathcal{A}$-modules, we define their sum $f + g$ by

$$(f + g)(U) = f(U) + g(U) : \mathcal{L}(U) \to \mathcal{M}(U)$$

for each open set $U \subset X$. With respect to this addition, the zero morphism is clear. This gives a structure of abelian group on the set $\mathrm{Hom}_{\mathcal{A}}(\mathcal{L}, \mathcal{M})$ of morphisms between $\mathcal{A}$-modules $\mathcal{L}$ and $\mathcal{M}$. The category of $\mathcal{A}$-modules is denoted by $\mathcal{A} - \mathsf{Mod}$.

The direct product and the direct sum of $\mathcal{A}$-modules are defined in the natural way. They give the product object and the coproduct object in the category $\mathcal{A} - \mathsf{Mod}$.

For a sheaf $\mathcal{A}$ of rings on X and $\mathcal{A}$-modules $\mathcal{L}$ and $\mathcal{M}$, we can define the **sheaf hom** $\mathcal{H}om_{\mathcal{A}}(\mathcal{L}, \mathcal{M})$ to be the presheaf of abelian groups on X given by

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{L}, \mathcal{M})(U) = \mathrm{Hom}_{\mathcal{A}|_U}(\mathcal{L}|_U, \mathcal{M}|_U)$$

for each open set $U \subset X$. It turns out that $\mathcal{H}om_{\mathcal{A}}(\mathcal{L}, \mathcal{M})$ is a sheaf of abelian groups on X, and if $\mathcal{A}$ is commutative, then $\mathcal{H}om_{\mathcal{A}}(\mathcal{L}, \mathcal{M})$ is canonically an $\mathcal{A}$-module.

Theorem 2.1 Suppose X is a topological space and $\mathcal{A}$ is a sheaf of commutative rings with identity on X. Then for each $\mathcal{A}$-module $\mathcal{M}$, the sheaf hom $\mathcal{H}om_{\mathcal{A}}(\mathcal{A}, \mathcal{M})$ is naturally isomorphic to $\mathcal{M}$ as an $\mathcal{A}$-module.

An **$\mathcal{A}$-submodule** and a **quotient $\mathcal{A}$-module** of an $\mathcal{A}$-module $\mathcal{M}$ on X are defined to be a subobject and a quotient object of $\mathcal{M}$ in the category $\mathcal{A} - \mathsf{Mod}$, respectively. However, we may have an alternative intuitive definition. A submodule of $\mathcal{M}$ is an $\mathcal{A}$-module $\mathcal{N}$ such that for each open set U, $\mathcal{N}(U)$ is an $\mathcal{A}(U)$-submodule of $\mathcal{M}(U)$, and that the restiction morphisms commute with the inclusion of $\mathcal{N}(U)$ into $\mathcal{M}(U)$. The quotient $\mathcal{A}$-module $\mathcal{M}/\mathcal{N}$ is then defined to be the sheaf of abelian groups associated to the presheaf $U \mapsto \mathcal{M}(U)/\mathcal{N}(U)$, with the structure of $\mathcal{A}$-module given by the induced action of $\mathcal{A}(U)$ on $\mathcal{M}(U)/\mathcal{N}(U)$. Then for each $x \in X$, the stalk $\mathcal{N}_x$ can be identified with an $\mathcal{A}_x$-submodule of $\mathcal{M}_x$, and the stalk $(\mathcal{M}/\mathcal{N})_x$ can be identified with the quotient $\mathcal{A}_x$-module $\mathcal{M}_x/\mathcal{N}_x$.

Theorem 2.2 Suppose X is a topological space and $\mathcal{A}$ is a sheaf of rings on X. Then the category $\mathcal{A} - \mathsf{Mod}$ of $\mathcal{A}$-modules is an abelian category.

Proof. Suppose $f : \mathcal{L} \to \mathcal{M}$ is a morphism of $\mathcal{A}$-modules. The presheaf $U \mapsto \ker f(U)$ is actually a sheaf, which is defined to be the kernel of f. The sheaves associated to the presheaves $U \mapsto \mathrm{im} f(U)$ and $U \mapsto \mathrm{coker} f(U)$, are define to be the image and cokernel of f, respectively. We can see that $\ker f$ is a $\mathcal{A}$-submodule of $\mathcal{L}$, $\mathrm{im} f$ is a $\mathcal{A}$-submodule of $\mathcal{M}$ and a quotient $\mathcal{A}$-module of $\mathcal{L}$, and $\mathrm{coker} f$ is a quotient $\mathcal{A}$-module of $\mathcal{M}$, all in a natural way. We also have the natural $\mathcal{A}$-module isomorphisms

$$\mathcal{L}/\ker f \xrightarrow{\sim} \mathrm{im} f, \quad \mathcal{M}/\mathrm{im} f \xrightarrow{\sim} \mathrm{coker} f.$$

□

A sequence of $\mathcal{A}$-modules

$$\mathcal{L} \xrightarrow{f} \mathcal{M} \xrightarrow{g} \mathcal{N}$$

is called an **exact sequence** if the image of f is equal to the kernel of g, just as the case of a general abelian category.

Theorem 2.3 Suppose X is a topological space and $\mathcal{A}$ is a sheaf of rings on X. Then the sequence of $\mathcal{A}$-modules

$$\mathcal{L} \xrightarrow{f} \mathcal{M} \xrightarrow{g} \mathcal{N}$$

is exact if and only if for each $x \in X$, the induced sequence of $\mathcal{A}_x$-modules

$$\mathcal{L}_x \xrightarrow{f_x} \mathcal{M}_x \xrightarrow{g_x} \mathcal{N}_x$$

is exact.

Remark This shows that the functor $\mathcal{A} - \mathsf{Mod} \to \mathcal{A}_x - \mathsf{Mod}$ given by

$$\mathcal{L} \mapsto \mathcal{L}_x$$

is exact for each $x \in X$. However, it worth noting that for an open subset $U \subset X$, the functor

$$\Gamma(U, -) : \mathcal{A} - \mathsf{Mod} \to \mathcal{A}(U) - \mathsf{Mod}$$

is only left exact.

Example 2.3 Suppose M is an n-dimesnional smooth manifold. For each $p \geq 0$, consider the $\underline{\mathbb{R}}$-module Ω^p of differential p-forms on M, where $\Omega^p(U)$ is the differential p-forms on U for each open set $U \subset M$. The exterior derivative d gives a morphism $\Omega^p \to \Omega^{p+1}$ of $\underline{\mathbb{R}}$-modules for each $p \geq 0$. Moreover, we can embed $\underline{\mathbb{R}}$ into Ω^0 by viewing a locally constant function as a differential 0-form, i.e, a smooth function on M. Then we obtain a sequence of $\underline{\mathbb{R}}$-modules

$$0 \longrightarrow \underline{\mathbb{R}} \longrightarrow \Omega^0 \xrightarrow{\mathrm{d}} \Omega^1 \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^n \longrightarrow 0$$

By Poincare lemma, we have the following exact sequence of $\mathbb{R}$-spaces:

$$0 \longrightarrow \mathbb{R} \longrightarrow \Omega^0(\mathbb{R}^n) \xrightarrow{\mathrm{d}} \Omega^1(\mathbb{R}^n) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^n(\mathbb{R}^n) \longrightarrow 0 \ ,$$

which implies the exact sequence of stalks at each $x \in M$ as M is locally homeomorphic to $\mathbb{R}^n$. Thus the above sequence of sheaves is an exact sequence of $\underline{\mathbb{R}}$-modules.

Suppose $\mathcal{A}$ is a sheaf of rings on X, $\mathcal{L}$ is a right $\mathcal{A}$-module and $\mathcal{M}$ is a left $\mathcal{A}$-module. Then we can define the **tensor product** $\mathcal{L} \otimes_{\mathcal{A}} \mathcal{M}$ to be the sheaf of abelian groups on X associated to the presheaf

$$U \mapsto \mathcal{L}(U) \otimes_{\mathcal{A}(U)} \mathcal{M}(U)$$

for each open set $U \subset X$. We can see that $(\mathcal{L} \otimes_{\mathcal{A}} \mathcal{M})_x$ is naturally isomorphic to the tensor product $\mathcal{L}_x \otimes_{\mathcal{A}_x} \mathcal{M}_x$ for each $x \in X$, and that each mid-linear morphism $\mathcal{L} \times \mathcal{M} \to \mathcal{N}$ of sheaves of abelian groups factors through the canonical mid-linear morphism $\mathcal{L} \times \mathcal{M} \to \mathcal{L} \otimes_{\mathcal{A}} \mathcal{M}$.

For a continuous map $f : X \to Y$ between topological spaces, we can also consider the direct image and inverse image of sheaves of modules. Suppose $\mathcal{A}$ is a sheaf of rings on X and $\mathcal{M}$ is an $\mathcal{A}$-modules. Then the direct image $f_*\mathcal{A}$ is a sheaf of rings on Y, and the direct image $f_*\mathcal{M}$, as a sheaf of abelian groups, is canonically a $f_*\mathcal{A}$-module. We then obtain a functor

$$f_* : \mathcal{A} - \mathsf{Mod} \to f_*\mathcal{A} - \mathsf{Mod}\,.$$

Dually, if $\mathcal{B}$ is a sheaf of rings on Y and $\mathcal{N}$ is a $\mathcal{B}$-module, then the inverse image $f^{-1}\mathcal{B}$ is a sheaf of rings on X, and the inverse image $f^{-1}\mathcal{N}$, as a sheaf of abelian groups, is canonically a $f^{-1}\mathcal{B}$-module. This yields another functor

$$f^{-1} : \mathcal{B} - \mathsf{Mod} \to f^{-1}\mathcal{B} - \mathsf{Mod}\,.$$

Theorem 2.4 Suppose $f : X \to Y$ is a continuous map, $\mathcal{A}$ is a sheaf of rings on X and $\mathcal{B}$ is a sheaf of rings on Y. Then the functor $f^{-1} : \mathcal{B} - \mathsf{Mod} \to f^{-1}\mathcal{B} - \mathsf{Mod}$ is exact, and the functor $f_* : \mathcal{A} - \mathsf{Mod} \to f_*\mathcal{A} - \mathsf{Mod}$ is left exact.

Definition 2.4 (family of support) Suppose X is a topological space. A **family of support** Φ on X is a nonempty collection of closed sets in X such that:

1. if $A, B \in \Phi$, then $A \cup B \in \Phi$;
2. if $A \in \Phi$ and B is a closed subset of A, then $B \in \Phi$.

Suppose X is a topological space, Φ is a family of support on X and $\mathcal{F}$ is a sheaf of abelian groups on X. For each $s \in \Gamma(\mathcal{F}) = \mathcal{F}(X)$, we define the **support** of s, denoted by $\mathrm{supp}(s)$, to the set of points $x \in X$ such that $s_x \neq 0$, where s_x is the corresponding element of s in the stalk $\mathcal{F}_x$. It is clear that $\mathrm{supp}(s)$ is always a closed set in X. Consider the subset $\Gamma_\Phi(\mathcal{F})$ of $\Gamma(\mathcal{F})$ given by

$$\Gamma_\Phi(\mathcal{F}) = \{s \in \Gamma(\mathcal{F}) \mid \mathrm{supp}(s) \in \Phi\}.$$

We can verify that $\Gamma_\Phi(\mathcal{F})$ is a subgroup of the abelian group $\Gamma(\mathcal{F})$.

Theorem 2.5 Suppose X is a topological space, Φ is a family of support on X. Then the map

$$\mathcal{F} \mapsto \Gamma_\Phi(\mathcal{F})$$

gives a left exact functor

$$\Gamma_\Phi : \mathsf{Sh}(X, \mathsf{Ab}) \to \mathsf{Ab}\,.$$

3 Extension and lifting of sections

3.1 Flasque sheaves

Definition 3.1 (flasque sheaf) A sheaf $\mathcal{F}$ on a topological space X is called **flasque** if for each open set $U \subset X$, the restriction morphism

$$\mathcal{F}(X) \to \mathcal{F}(U)$$

is surjective.

Proposition 3.1 Suppose X is a topological space and $\mathcal{F}$ is a sheaf on X. If for each $x \in X$, there is a neighborhood U of X such that $\mathcal{F}|_U$ is flasque, then $\mathcal{F}$ is flasque.

Proof. Fix an open set $U \subset X$ and a section $s \in \mathcal{F}(U)$. Consider the poset

$$E = \{(U', s') \mid U \subset U' \text{ and } s = s'|_U\},$$

with the partial order given by

$$(U', s') \leq (U'', s'') \quad \Longleftrightarrow \quad U' \subset U'' \text{ and } s' = s''|_{U'}.$$

Then E is a nonempty collection satisfying the condition of Zorn's lemma, and then we can find a maximal element $(\tilde{U}, \tilde{s}) \in E$. We claim that $\tilde{U} = X$. Otherwise, there exists a neighborhood V of $x \in X \setminus \tilde{U}$ such that $\mathcal{F}|_V$ is flasque. Then we can find a section $\bar{s} \in \mathcal{F}(V)$ such that $\bar{s}|_{\tilde{U}\cap V} = \tilde{s}|_{\tilde{U}\cap V}$. The axiom of sheaf yields a section $\tilde{s}' \in \mathcal{F}(\tilde{U} \cup V)$ such that $\tilde{s}'|_{\tilde{U}} = \tilde{s}$, contradicting the maximality of $(\tilde{U}, \tilde{s})$. Thus we have $\tilde{U} = X$, and then $\tilde{s} \in \mathcal{F}(X)$, which shows that $\mathcal{F}$ is flasque. □

Remark For an element $(U', s') \in E$, we usually call s' a **extension** of s to U'.

By the definition of direct image, we have the following result directly.

Proposition 3.2 Suppose X and Y are topological spaces, $f : X \to Y$ is a continuous map and $\mathcal{F}$ is a sheaf on X. If $\mathcal{F}$ is flasque, then the direct image sheaf $f_*\mathcal{F}$ is also flasque.

An important result of flasque sheaves is about the exact sequences.

Theorem 3.3 Suppose X is a topological space and

$$0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0$$

is an exact sequence of sheaves of abelian groups on X. If $\mathcal{L}$ is flasque, then the induced sequence

$$0 \to \mathcal{L}(U) \to \mathcal{M}(U) \to \mathcal{N}(U) \to 0$$

is exact for each open set $U \subset X$.

Proof. Suppose the morphisms of sheaves are $f : \mathcal{L} \to \mathcal{M}$ and $g : \mathcal{M} \to \mathcal{N}$, with induced morphisms $\tilde{f}$ and $\tilde{g}$. As long as $\Gamma(U, -)$ is left exact,it suffices to show that the morphism $\mathcal{M}(U) \to \mathcal{N}(U)$ is surjective. Fix a section $t \in \mathcal{N}(U)$. Consider the poset

$$E = \{(V, s) \mid V \subset U \text{ and } \tilde{g}(s) = t|_V\}.$$

The surjectivity of the sheaf morphism $\mathcal{M} \to \mathcal{N}$ implies that E is nonempty, and it can be verified that E satisfies the condition of Zorn's lemma. Thus there is a maximal element $(\tilde{V}, \tilde{s}) \in E$. We claim that $\tilde{V} = U$. Otherwise, take $x \in U \setminus \tilde{V}$. Since $\mathcal{M}_x \to \mathcal{N}_x$ is surjective, there exists a section $\bar{s}$ on a neighborhood W of x such that $\tilde{g}(\bar{s}) = t|_W$. Then

$$\tilde{g}(\tilde{s}|_{\tilde{V}\cap W} - \bar{s}|_{\tilde{V}\cap W}) = 0,$$

i.e.

$$\tilde{s}|_{\tilde{V}\cap W} - \bar{s}|_{\tilde{V}\cap W} \in \ker \tilde{g} = \operatorname{im} \tilde{f}.$$

Suppose

$$\tilde{s}|_{\tilde{V}\cap W} - \bar{s}|_{\tilde{V}\cap W} = \tilde{f}(u)$$

for some $u \in \mathcal{L}(\tilde{V} \cap W)$. Since $\mathcal{L}$ is flasque, we can extend u to a section $\tilde{u} \in \mathcal{L}(W)$. Then $\tilde{s} \in \mathcal{M}(\tilde{V})$ and $\bar{s} + \tilde{f}(\tilde{u}) \in \mathcal{M}(W)$ agree on $\tilde{V} \cap W$, which induces a section $\tilde{s}' \in \mathcal{M}(\tilde{V} \cup W)$ such that $\tilde{s}'|_{\tilde{V}} = \tilde{s}$. This contradicts the maximality of $(\tilde{V}, \tilde{s})$. Thus we have $\tilde{V} = U$, and hence $\tilde{g}(\tilde{s}) = t$. □

Remark The section $\tilde{s}$ is usually called a **lifting** of t to $\mathcal{M}$.

Corollary 3.4 Suppose X is a topological space and

$$0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0$$

is an exact sequence of sheaves of abelian groups on X. If $\mathcal{L}$ and $\mathcal{M}$ are flasque, then $\mathcal{N}$ is also flasque.

Proof. For each open set $U \subset X$, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L}(X) & \longrightarrow & \mathcal{M}(X) & \longrightarrow & \mathcal{N}(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{L}(U) & \longrightarrow & \mathcal{M}(U) & \longrightarrow & \mathcal{N}(U) & \longrightarrow & 0 \end{array}$$

In particular, Since $\mathcal{M}(X) \to \mathcal{M}(U)$ and $\mathcal{M}(U) \to \mathcal{N}(U)$ are surjective, their composition $\mathcal{M}(X) \to \mathcal{N}(U)$ is also surjective. The surjectivity of $\mathcal{N}(X) \to \mathcal{N}(U)$ then follows. □

Theorem 3.5 Suppose

$$0 \to \mathcal{M}^0 \to \mathcal{M}^1 \to \mathcal{M}^2 \to \cdots$$

is an exact sequence of flasque sheaves of abelian groups on a topological space X. Then for each family Φ of support on X, the sequence

$$0 \to \Gamma_\Phi(\mathcal{M}^0) \to \Gamma_\Phi(\mathcal{M}^1) \to \Gamma_\Phi(\mathcal{M}^2) \to \cdots$$

of abelian groups is exact.

Proof. Let

$$\mathcal{Z}^p = \ker(\mathcal{M}^p \to \mathcal{M}^{p+1}) = \operatorname{im}(\mathcal{M}^{p-1} \to \mathcal{M}^p)$$

for each $p \geq 0$. Since we have the exact sequence

$$0 \to \mathcal{Z}^p \to \mathcal{M}^p \to \mathcal{M}^{p+1}$$

and Γ_Φ is left exact, the sequence

$$0 \to \Gamma_\Phi(\mathcal{Z}^p) \to \Gamma_\Phi(\mathcal{M}^p) \to \Gamma_\Phi(\mathcal{M}^{p+1})$$

is exact. Thus it suffices to show that

$$0 \to \Gamma_\Phi(\mathcal{Z}^p) \to \Gamma_\Phi(\mathcal{M}^p) \to \Gamma_\Phi(\mathcal{Z}^{p+1}) \to 0$$

is exact for each $p \geq 0$. We consider the exact sequence

$$0 \to \mathcal{Z}^p \to \mathcal{M}^p \to \mathcal{Z}^{p+1} \to 0.$$

As Γ_Φ is left exact, we only need to show that $\Gamma_\Phi(\mathcal{M}^p) \to \Gamma_\Phi(\mathcal{Z}^{p+1})$ is surjective. Since $\mathcal{M}^p$ is flasque, the flasque property of $\mathcal{Z}^p$ implies the flasque property of $\mathcal{Z}^{p+1}$. As long as $\mathcal{Z}^0 = 0$ is flasque, the induction on p shows that each $\mathcal{Z}^p$ is flasque. Thus $\Gamma(\mathcal{M}^p) \to \Gamma(\mathcal{Z}^{p+1})$ is surjective. Now take any $t \in \Gamma_\Phi(\mathcal{Z}^{p+1})$, with $\operatorname{supp}(t) = S \in \Phi$. We can lift t to a section $s \in \Gamma(\mathcal{M}^p)$, whose support is not necessarily in Φ. However, $s|_{X \setminus S}$ maps to zero in $\mathcal{Z}^{p+1}$, implying that $s|_{X \setminus S}$ is a section of $\mathcal{Z}^p$. Since $\mathcal{Z}^p$ is flasque, we can extend $s|_{X \setminus S}$ to a section $s' \in \Gamma(\mathcal{Z}^p)$. Then $s - s'$ is a section of $\mathcal{M}^p$ with support contained in S, and hence

$$s - s' \in \Gamma_\Phi(\mathcal{M}^p).$$

It is clear that $s - s'$ is a lifting of t in $\Gamma_\Phi(\mathcal{M}^p)$. □

3.2 Paracompactified family and soft sheaves

A topological space X is called **paracompact** if every open cover of X has a locally finite open refinement. A closed subspace of a paracompact space is also paracompact. A paracompact Hausdorff space is normal.

Lemma 3.6 Suppose X is a paracompact Hausdorff space. If $\{U_i\}_{i \in I}$ is an open cover of X, then there exists a locally finite open refinement $\{V_i\}_{i \in I}$ such that $V_i \subset U_i$ for each $i \in I$.

Lemma 3.7 Suppose X is a normal space. If $\{U_i\}_{i \in I}$ is a locally finite open cover of X, then there exists another locally finite open cover $\{V_i\}_{i \in I}$ such that $\overline{V}_i \subset U_i$ for each $i \in I$.

Corollary 3.8 Suppose X is a paracompact Hausdorff space. Then each open cover $\{U_i\}_{i\in I}$ of X has a locally finite open refinement $\{V_i\}_{i\in I}$ such that $\overline{V}_i \subset U_i$ for each $i \in I$.

Definition 3.2 (paracompactified family) Suppose X is a topological space. A **paracompactified family** Φ on X is a nonempty family of closed subsets of X such that:

1. each $S \in \Phi$ is paracompact and Hausdorff;
2. if $S_1, \cdots, S_n \in \Phi$, then $S_1 \cup \cdots \cup S_n \in \Phi$;
3. if $S \in \Phi$ and $S' \subset S$ is a closed subset, then $S' \in \Phi$;
4. each $S \in \Phi$ has a neighborhood U such that $\overline{U} \in \Phi$.

If Φ is a family of support on X and Y is a subspace of Y, then define $\Phi|_Y$ to be the family of subsets $S \in \Phi$ such that $S \subset Y$. If Y is a closed subset of X, then

$$\Phi|_Y = \{S \cap Y \mid S \in \Phi\}.$$

If Φ is paracompactified and $Y = U \cap F$ with U open and F closed in X, then we can verify that $\Phi|_Y$ is a paracompactified family on Y.

Suppose $\mathcal{F}$ is a sheaf on a topological space X. Then we can actually define sections of $\mathcal{F}$ on any subset Y of X to be continuous maps $s : Y \to E$ such that $p \circ s$ is the identity map on Y, where (E, p) is the etale space of $\mathcal{F}$. The restriction morphisms are defined in the natural way.

Theorem 3.9 Suppose $\mathcal{F}$ is a sheaf on a topological space X and $\{Y_i\}_{i\in I}$ is a locally finite closed cover of X. If $s_i \in \mathcal{F}(Y_i)$ are sections such that

$$s_i|_{Y_i\cap Y_j} = s_j|_{Y_i\cap Y_j}, \quad i, j \in I,$$

then there exists a section $s \in \mathcal{F}(X)$ such that $s|_{Y_i} = s_i$ for each $i \in I$.

Proof. Suppose (E, p) is the etale space of $\mathcal{F}$. It is direct that there exists a map $s : X \to E$ such that $p \circ s$ is identity on X and $s|_{Y_i} = s_i$ for each $i \in I$. It remains to show that s is continuous. Fix a point $x \in X$. Since $\{Y_i\}_{i\in I}$ is locally finite, there exists a neighborhood U of x such that $U \cap Y_i$ is nonempty for only finitely many $i_1, \cdots, i_n \in I$. By shrinking U if necessary, we may assume that x is contained in each Y_{i_k}, and that there exists a section t of $\mathcal{F}$ on U such that

$$t(x) = s(x) = s_{i_1}(x) = \cdots = s_{i_n}(x).$$

For each $1 \le k \le n$, there exists a neighborhood U_k of x such that t and s_{i_k} agree on $U_k \cap Y_{i_k}$. Let $U' = U_1 \cap \cdots U_n$. Then t agrees with s on each $U' \cap Y_{i_k}$, and hence on U'. The continuity of s at x follows. □

Theorem 3.10 Suppose $\mathcal{F}$ is a sheaf on a topological space X, S is a subset of X and s is a section of $\mathcal{F}$ on S. If S has a fundamental system of neighborhoods consisting of paracompact Hausdorff subsets in X, then s can be extended to a neighborhood of S in X.

Proof. Try to use Theorem 3.9 to glue the sections. □

Corollary 3.11 Suppose X is a topological space, $\mathcal{F}$ is a sheaf on X and S is a subset of X with a fundamental system of neighborhoods consisting of paracompact Hausdorff subsets in X. Then we have

$$\mathcal{F}(S) = \varinjlim \mathcal{F}(U),$$

where the inductive limit is taken over open neighborhoods U of S in X.

It follows from the above corollary that if X is a paracompact Hausdorff space and $\mathcal{F}$ is a flasque sheaf on X, then each section of $\mathcal{F}$ on a closed subset of X can be extended to the whole space X.

Definition 3.3 (soft sheaf) A sheaf $\mathcal{F}$ on a topological space X is called **soft** if for each closed subset S of X, each section of $\mathcal{F}$ on S can be extended to X.

It is direct that the restriction of a soft sheaf to a closed subset is also soft.

Theorem 3.12 Suppose $\mathcal{F}$ is a sheaf on a paracompact Hausdorff space X. Suppose that for each $x \in X$, there exists a neighborhood U of x such that each section of $\mathcal{F}$ on a subset of U closed in X can extended to U. Then $\mathcal{F}$ is soft.

Proof. Suppose s is a section of $\mathcal{F}$ on a closed subset S of X. Since X is paracompact Hausdorff, we can take a locally finite open cover $\{U_i\}_{i\in I}$ of X such that each U_i satisfies the extention property stated in the theorem. Then there exists another locally finite open cover $\{V_i\}_{i\in I}$ of X such that

$$F_i := \overline{V}_i \subset U_i$$

for each $i \in I$. For each $J \subset I$, define

$$F_J = \bigcup_{i\in J} F_i.$$

Now consider the poset

$$E = \{(J,t) \mid J \subset I, \text{ and } t \text{ is a section of } \mathcal{F} \text{ on } F_J \text{ such that } t|_{S\cap F_J} = s|_{S\cap F_J}\},$$

with the partial order given by

$$(J,t) \le (J',t') \quad \Longleftrightarrow \quad J \subset J', \text{ and } t = t'|_{F_J}.$$

The extension property on each U_i implies that E is nonempty, and Theorem 3.9 shows that E satisfies the condition of Zorn's lemma. Thus we can find a maximal element $(\tilde{J},\tilde{t}) \in E$. We claim that $\tilde{J} = I$. Otherwise there exists $i \in I \setminus \tilde{J}$. Let $\tilde{J}' = \tilde{J} \cup \{i\}$. Noting that $s|_{S\cap F_i}$ is a section of $\mathcal{F}$ on $S \cap F_i$, which is a subset of U_i closed in X, we can extend this to a section s' of $\mathcal{F}$ on F_i by the choice of U_i. Theorem 3.9 then implies that there exists a section $\tilde{t}'$ on $F_{\tilde{J}'}$ agreeing with $\tilde{t}$ on $F_{\tilde{J}}$ and with s on $S \cap F_{\tilde{J}'}$. This contradicts the maximality of $(\tilde{J},\tilde{t})$. Thus we have $\tilde{J} = I$, and hence $\tilde{t}$ is a section of $\mathcal{F}$ on X such that $\tilde{t}|_S = s$. □

Corollary 3.13 Suppose X is a paracompact Hausdorff space and $\{\mathcal{F}_i\}_{i\in I}$ is a locally finite family of sheaves of abelian groups on X. If each $\mathcal{F}_i$ is soft, then their direct sum is also soft.

Proof. The statement is trivial for finite I. We then use the local finiteness and Theorem 3.12 to deal with a general I. □

Definition 3.4 (Φ-soft sheaf) Suppose X is a topological space and Φ is a paracompactified family on X. A sheaf $\mathcal{F}$ on X is called **Φ-soft** if for each $S \in \Phi$, $\mathcal{F}|_S$ is soft, i.e., for each $S', S \in \Phi$ with $S' \subset S$, the restriction morphism $\mathcal{F}(S) \to \mathcal{F}(S')$ is surjective.

Theorem 3.14 Suppose X is a paracompact Hausdorff space, Φ is a paracompactified family on X, and $\mathcal{F}$ is a sheaf of abelian groups on X. Then $\mathcal{F}$ is Φ-soft if and only if for each $S \in \Phi$, the morphism $\Gamma_\Phi(\mathcal{F}) \to \mathcal{F}(S)$ is surjective.

Proof. If the corresponding morphism is surjective for each $S \in \Phi$, then it is direct that $\mathcal{F}$ is Φ-soft. Conversely, suppose $\mathcal{F}$ is Φ-soft. For each $S \in \Phi$, we can find a neighborhood U of S such that $\overline{U} \in \Phi$. There is a section on $S \cup (\overline{U} \setminus U)$ given by s on S and 0 on $\overline{U} \setminus U$. By the Φ-soft property, we can extend this to a section $\tilde{s}$ on $\overline{U}$. Taking zero in $X \setminus \overline{U}$, $\tilde{s}$ then extends to the whole space X. It can be seen that the extension belongs to $\Gamma_\Phi(\mathcal{F})$. □

Theorem 3.15 Suppose X is topological space, Φ is a paracompactified family on X, and

$$0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0$$

is an exact sequence of sheaves of abelian groups on X. If $\mathcal{L}$ is Φ-soft, then

$$0 \to \Gamma_\Phi(\mathcal{L}) \to \Gamma_\Phi(\mathcal{M}) \to \Gamma_\Phi(\mathcal{N}) \to 0$$

is an exact sequence of abelian groups.

Proof. First suppose X is paracompact Hausdorff and Φ consists of all closed subsets of X. Using arguments similar to the proofs of Theorem 3.3 and Theorem 3.12, we see that each section of $\mathcal{N}$ on X can be lifted to a section of $\mathcal{M}$. For a general X and Φ, just consider the support of the section needed to be lifted. □

Corollary 3.16 Suppose X is a paracompact Hausdorff space and

$$0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0$$

is an exact sequence of sheaves of abelian groups on X. If $\mathcal{L}$ is soft, then the sequence

$$0 \to \mathcal{L}(A) \to \mathcal{M}(A) \to \mathcal{N}(A) \to 0$$

is exact for each closed subset A of X.

Analogous to the cases of flasque sheaves, we have the follwing theorems.

Theorem 3.17 Suppose X is a topoogical space, Φ is a paracompactified family on X, and

$$0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0$$

is an exact sequence of sheaves of abelian groups on X. If $\mathcal{L}$ and $\mathcal{M}$ are Φ-soft, then $\mathcal{N}$ is also Φ-soft.

Theorem 3.18 Suppose X is a topological space, Φ is a paracompactified family on X, and

$$0 \to \mathcal{M}^0 \to \mathcal{M}^1 \to \mathcal{M}^2 \to \cdots$$

is an exact sequence of Φ-soft sheaves of abelian groups on X. Then the sequence

$$0 \to \Gamma_\Phi(\mathcal{M}^0) \to \Gamma_\Phi(\mathcal{M}^1) \to \Gamma_\Phi(\mathcal{M}^2) \to \cdots$$

is exact.

A useful result of soft sheaves is the following theorem.

Theorem 3.19 Suppose X is a topological space, Φ is a paracompactified family on X, and $\mathcal{A}$ is a sheaf of rings with identity on X. If $\mathcal{A}$ is Φ-soft, then each $\mathcal{A}$-module $\mathcal{M}$ is also Φ-soft.

Proof. Suppose s is a section of $\mathcal{M}$ on a closed subset $S \in \Phi$ of X. There exists a neighborhood U of S such that $\overline{U} \in \Phi$. Since $\mathcal{A}$ is Φ-soft, we can take a section u of $\mathcal{A}$ on $\overline{U}$ such that u takes the value 1 on S and 0 on $\overline{U} \setminus U$. Taking zero in $X \setminus \overline{U}$, we can extend the section

$$x \mapsto u(x)s(x)$$

of $\mathcal{M}$ on $\overline{U}$ to the whole space X. □

4 Sheaf cohomology

A sheaf is always a sheaf of abelian groups without stated in this section.

4.1 Cohomology sheaf of a differential sheaf

Definition 4.1 (graded sheaf) Suppose X is a topological space. A **graded sheaf** on X is a sequence $\mathcal{F}^* = \{\mathcal{F}^n\}_{n\in\mathbb{Z}}$ of sheaves on X, where $\mathcal{F}^n$ is called the **component of degree** n of $\mathcal{F}^*$.

Suppose T is a functor from $\mathsf{Sh}(X, \mathsf{Ab})$ to Ab (or generally any abelian category), denote by $T(\mathcal{F}^*)$ the graded abelian group $\{T(\mathcal{F}^n)\}_{n\in\mathbb{Z}}$. It is worth noting that $T(\mathcal{F}^*)$ is not necessarily identified canonically with $T(\bigoplus \mathcal{F}^n)$.

For two graded sheaf $\mathcal{F}^*$ and $\mathcal{G}^*$ on X, a **homomorphism of degree** r from $\mathcal{F}^*$ to $\mathcal{G}^*$ is a sequence $f = \{f^n\}_{n\in\mathbb{Z}}$ of morphisms $f^n : \mathcal{F}^n \to \mathcal{G}^{n+r}$. When $r = 0$, we simply call it a hoomorphism from $\mathcal{F}^*$ to $\mathcal{G}^*$. It can be verified that the graded sheaves on X, together with the homomorphisms, form an abelian category.

Definition 4.2 (differential sheaf) A **differential sheaf** on X is a graded sheaf $\mathcal{F}^*$ together with a homomorphism $d : \mathcal{F}^* \to \mathcal{F}^*$ of degree r, satisfying $d^2 = 0$. We are mostly concerned with the case $r = 1$.

A homomorphism of differential sheaves is a homomorphism of graded sheaves commuting with the differentials. We can also verify that the differential sheaves on X also form an abelian category.

Suppose $\mathcal{F}^*$ is a differential sheaf on X. Define that

$$\mathcal{Z}^n(\mathcal{F}^*) = \ker(\mathcal{F}^n \xrightarrow{d} \mathcal{F}^{n+1}), \quad \mathcal{B}^n(\mathcal{F}^*) = \operatorname{im}(\mathcal{F}^{n-1} \xrightarrow{d} \mathcal{F}^n), \quad \mathcal{H}^n(\mathcal{F}^*) = \mathcal{Z}^n(\mathcal{F}^*)/\mathcal{B}^n(\mathcal{F}^*).$$

The sheaf $\mathcal{H}^n(\mathcal{F}^*)$ is called the **derived sheaf** (of degree n) of $\mathcal{F}^*$.

It is noticeable that the concept of differential sheaves and derived sheaves are analogous to cochain complexes and homology groups. Suppose T is an additive functor from $\mathsf{Sh}(X, \mathsf{Ab})$ to Ab. Then for each differential sheaf $\mathcal{F}^*$, the graded group $T(\mathcal{F}^*)$, together with the differential $T(d) : T(\mathcal{F}^n) \to T(\mathcal{F}^{n+1})$, is a cochain complex. If T is left exact, then consider the exact sequence

$$0 \to \mathcal{Z}^n \to \mathcal{F}^n \xrightarrow{d} \mathcal{F}^{n+1},$$

which yields the exact sequence

$$0 \to T(\mathcal{Z}^n) \to T(\mathcal{F}^n) \to T(\mathcal{F}^{n+1}).$$

We then identity $T(\mathcal{Z}^n)$ with $Z^n(T(\mathcal{F}^*))$ in a canonical way. If we further have T is exact, then using the exact sequences

$$0 \to \mathcal{Z}^n \to \mathcal{F}^n \to \mathcal{B}^{n+1} \to 0$$

and

$$0 \to \mathcal{B}^n \to \mathcal{Z}^n \to \mathcal{H}^n \to 0,$$

we obtain a canonical isomorphism

$$H^n(T(\mathcal{F}^*)) = T(\mathcal{H}^n(\mathcal{F}^*)), \quad n \in \mathbb{Z}.$$

For instance, for each $x \in X$, we have the canonical isomorphism

$$H^n(\mathcal{F}_x^*) = (\mathcal{H}^n(\mathcal{F}^*))_x, \quad n \in \mathbb{Z}.$$

An equivalent definition of the derived sheaves of a differential sheaf $\mathcal{F}$ is as follows. We can see that the sheaves $\mathcal{Z}^n$ and $\mathcal{B}^n$ are generated by the presheaves

$$U \mapsto Z^n(\mathcal{F}^*(U)), \quad U \mapsto B^n(\mathcal{F}^*(U)),$$

respectively. Then the derived sheaf $\mathcal{H}^n(\mathcal{F}^*)$ of degree n is generated by the presheaf

$$U \mapsto H^n(\mathcal{F}^*(U)).$$

4.2 Resolution

Now we construct a differential sheaf from a sheaf to define the cohomology of sheaves.

Definition 4.3 (resolution) Suppose X is a topological space and $\mathcal{A}$ is a sheaf on X. A (cohomological) **resolution** is an exact sequence of sheaves

$$0 \to \mathcal{A} \xrightarrow{j} \mathcal{F}^0 \xrightarrow{d} \mathcal{F}^1 \xrightarrow{d} \mathcal{F}^2 \to \cdots.$$

The associated differential sheaf of resolution is $\mathcal{F}^* = \{\mathcal{F}^n\}_{n\in\mathbb{Z}}$, with $\mathcal{F}^n = 0$ for $n < 0$. We also call $\mathcal{F}^*$ a resolution of $\mathcal{A}$.

By definition, we see that the derived sheaf of a resolution of $\mathcal{A}$ is given by

$$\mathcal{H}^0(\mathcal{F}^*) = \mathcal{A}; \quad \mathcal{H}^n(\mathcal{F}^*) = 0 \text{ for } n > 0.$$

If T is an additive functor from $\mathsf{Sh}(X, \mathsf{Ab})$ to Ab, then $T(\mathcal{F}^*)$ is a cochain complex, with $T(\mathcal{A})$ embedded canonically into $H^0(T(\mathcal{F}^*))$. If T is left exact, then $T(\mathcal{A})$ is actually isomorphic to $H^0(T(\mathcal{F}^*))$; and if T is further exact, then $T(\mathcal{F}^*)$ is a resolution of $T(\mathcal{A})$.

Suppose $\mathcal{A}$ and $\mathcal{B}$ are two sheaves on X with resolutions $\mathcal{F}^*$ and $\mathcal{G}^*$, respectively. Suppose $f : \mathcal{A} \to \mathcal{B}$ is a morphism of sheaves. Then a homomorphism $g : \mathcal{F}^* \to \mathcal{G}^*$ of differential sheaves is said to be compatible with f if the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{F}^0 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathcal{B} & \longrightarrow & \mathcal{G}^0 \end{array}$$

For a sheaf $\mathcal{A}$ on X, we can constuct a resolution $\mathcal{C}^*(X; \mathcal{A})$ of $\mathcal{A}$ in a canonical way called the Godement resolution. First define $\mathcal{C}^0(X; \mathcal{A})$ to be the sheaf given by

$$U \mapsto \{s : U \to E \mid p(s(x)) = x \text{ for all } x \in U\},$$

where (E, p) is the etale space of $\mathcal{A}$ and the sections are not necessarily continuous. It is clear that $\mathcal{C}^0(X; \mathcal{A})$ is a flasque sheaf of abelian groups on X, together with a canonical embedding

$$j : \mathcal{A} \to \mathcal{C}^0(X; \mathcal{A}).$$

Next we define

$$\mathcal{Z}^1(X; \mathcal{A}) = \mathcal{C}^0(X; \mathcal{A})/\mathcal{A},$$

and then

$$\mathcal{C}^1(X; \mathcal{A}) = \mathcal{C}^0(X; \mathcal{Z}^1(X; \mathcal{A})).$$

This yields another embedding

$$\mathcal{Z}^1(X; \mathcal{A}) \to \mathcal{C}^1(X; \mathcal{A}).$$

For a general $n \geq 0$, suppose we have defined the sheaves $\mathcal{Z}^n(X; \mathcal{A})$ and $\mathcal{C}^n(X; \mathcal{A})$, with the former embedded in the latter, then we can define

$$\mathcal{Z}^{n+1}(X; \mathcal{A}) = \mathcal{C}^n(X; \mathcal{A})/\mathcal{Z}^n(X; \mathcal{A}), \quad \mathcal{C}^{n+1}(X; \mathcal{A}) = \mathcal{C}^0(X; \mathcal{Z}^{n+1}(X; \mathcal{A})).$$

This gives us the embedding

$$\mathcal{Z}^{n+1}(X; \mathcal{A}) \to \mathcal{C}^{n+1}(X; \mathcal{A}).$$

We can see that the sheaves $\mathcal{C}^n(X; \mathcal{A})$, $n \geq 0$ are all flasque.

The differential needs to be defined as well. It is quite direct to define

$$d : \mathcal{C}^n(X; \mathcal{A}) \to \mathcal{C}^{n+1}(X; \mathcal{A})$$

to be the composition

$$\mathcal{C}^n(X; \mathcal{A}) \to \mathcal{C}^n(X; \mathcal{A})/\mathcal{Z}^n(X; \mathcal{A}) = \mathcal{Z}^{n+1}(X; \mathcal{A}) \to \mathcal{C}^{n+1}(X; \mathcal{A}).$$

This verifies that d is a differential, and that the sequence

$$0 \to \mathcal{A} \xrightarrow{j} \mathcal{C}^0(X; \mathcal{A}) \xrightarrow{d} \mathcal{C}^1(X; \mathcal{A}) \xrightarrow{d} \cdots$$

is exact. Hence $\mathcal{C}^*(X; \mathcal{A})$ is a resolution of $\mathcal{A}$ by flasque sheaves, in other words, a flasque resolution of $\mathcal{A}$.

Let

$$C^*(X; \mathcal{A}) = \Gamma(\mathcal{C}^*(X; \mathcal{A})), \quad C^*_\Phi(X; \mathcal{A}) = \Gamma_\Phi(\mathcal{C}^*(X; \mathcal{A})),$$

where Φ is a family of support on X.

Theorem 4.1 Suppose X is a topological space, and Φ is a family of support on X. Then the assignments

$$\mathcal{A} \mapsto \mathcal{C}^*(X; \mathcal{A}), \quad \mathcal{A} \mapsto C^*_\Phi(X; \mathcal{A})$$

give exact additive functors from the category of sheaves to the category of differential sheaves and the cochain complexes, respectively. In particular, the functor given by

$$\mathcal{A} \mapsto C^*(X; \mathcal{A})$$

is exact.

Proof. Consider the exact sequence of sheaves

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0.$$

For each open set $U \subset X$, we have the exact sequence

$$0 \to \prod_{x \in U} \mathcal{A}_x \to \prod_{x \in U} \mathcal{B}_x \to \prod_{x \in U} \mathcal{C}_x \to 0,$$

which implies the exactness

$$0 \to \mathcal{C}^0(X;\mathcal{A}) \to \mathcal{C}^0(X;\mathcal{B}) \to \mathcal{C}^0(X;\mathcal{C}) \to 0.$$

Noting that we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{C} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{C}^0(X;\mathcal{A}) & \longrightarrow & \mathcal{C}^0(X;\mathcal{B}) & \longrightarrow & \mathcal{C}^0(X;\mathcal{C}) & \longrightarrow & 0 \end{array}$$

and that the morphism $\mathcal{C} \to \mathcal{C}^0(X;\mathcal{C})$ is injective, the snake lemma yields the exact sequence

$$0 \to \mathcal{Z}^1(X;\mathcal{A}) \to \mathcal{Z}^1(X;\mathcal{B}) \to \mathcal{Z}^1(X;\mathcal{C}) \to 0.$$

Doing this inductively, we obtain the exactness of the functor $\mathcal{C}^*(X;-)$. Theorem 3.5 then suggests that the functor $C^*_\Phi(X;-)$ is also exact. □

4.3 Cohomology groups of a sheaf

Definition 4.4 (cohomology of a sheaf) Suppose X is a topological space, Φ is a family of support on X and $\mathcal{A}$ is a sheaf on X. The **cohomology group** (of degree n) of $\mathcal{A}$ with respect to Φ is defined to be

$$H^n_\Phi(X;\mathcal{A}) = H^n(C^*_\Phi(X;\mathcal{A})) = H^n(\Gamma_\Phi(\mathcal{C}^*(X;\mathcal{A}))).$$

In particular, when Φ consists of all closed subsets of X, let

$$H^n(X;\mathcal{A}) = H^n(C^*(X;\mathcal{A})) = H^n(\Gamma(\mathcal{C}^*(X;\mathcal{A}))).$$

For a morphism $f : \mathcal{A} \to \mathcal{B}$, a homomorphism of groups

$$f^* : H^n_\Phi(X;\mathcal{A}) \to H^n_\Phi(X;\mathcal{B})$$

is induced for each $n \geq 0$. We then see that $H^n_\Phi(X;-)$ is a functor for each n.

Proposition 4.2 Suppose X is a topological space and Φ is a family of support on X. Then the functors

$$\Gamma_\Phi : \mathsf{Sh}(X,\mathsf{Ab}) \to \mathsf{Ab}, \quad H^0_\Phi(X,-) : \mathsf{Sh}(X,\mathsf{Ab}) \to \mathsf{Ab}$$

are naturally isomorphic.

Proof. For a sheaf $\mathcal{A}$, consider the exact sequence

$$0 \to \mathcal{A} \to \mathcal{C}^0(X;\mathcal{A}) \to \mathcal{C}^1(X;\mathcal{A}).$$

Since Γ_Φ is left exact, we have another exact sequence

$$0 \to \Gamma_\Phi(\mathcal{A}) \to C^0_\Phi(X;\mathcal{A}) \to C^1_\Phi(X;\mathcal{A}),$$

which implies the desired natural isomorphism. □

Theorem 4.3 Suppose X is a topological space, Φ is a family of support on X, and

$$0 \to \mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C} \to 0$$

is an exact sequence of sheaves on X. Then we have the long exact sequence

$$0 \to \Gamma_\Phi(\mathcal{A}) \xrightarrow{f^*} \Gamma_\Phi(\mathcal{B}) \xrightarrow{g^*} \Gamma_\Phi(\mathcal{C}) \xrightarrow{\partial} H^1_\Phi(X;\mathcal{A}) \xrightarrow{f^*} H^1_\Phi(X;\mathcal{B}) \xrightarrow{g^*} H^1_\Phi(X;\mathcal{C}) \to \cdots$$
$$\cdots \to H^{n-1}_\Phi(X;\mathcal{C}) \xrightarrow{\partial} H^n_\Phi(X;\mathcal{A}) \xrightarrow{f^*} H^n_\Phi(X;\mathcal{B}) \xrightarrow{g^*} H^n_\Phi(X;\mathcal{C}) \xrightarrow{\partial} H^{n+1}_\Phi(X;\mathcal{A}) \to \cdots.$$

Moreover, the connecting homorphism

$$\partial : H^n_\Phi(X;\mathcal{C}) \to H^{n+1}_\Phi(X;\mathcal{A})$$

is natural, in the sense that if we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{C} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{A}' & \longrightarrow & \mathcal{B}' & \longrightarrow & \mathcal{C}' & \longrightarrow & 0 \end{array}$$

then the following diagram commutes

$$\begin{array}{ccc} H^n_\Phi(X;\mathcal{C}) & \xrightarrow{\partial} & H^{n+1}_\Phi(X;\mathcal{A}) \\ \downarrow & & \downarrow \\ H^n_\Phi(X;\mathcal{C}') & \xrightarrow{\partial'} & H^{n+1}_\Phi(X;\mathcal{A}') \end{array}$$

Proof. Just consider the exact sequence of cochain complexes

$$0 \to C^*_\Phi(X;\mathcal{A}) \to C^*_\Phi(X;\mathcal{B}) \to C^*_\Phi(X;\mathcal{C}) \to 0.$$

□

Corollary 4.4 Suppose X is a topological space, Φ is a family of support on X, and

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

is an exact sequence of sheaves on X. If $H^1_\Phi(X;\mathcal{A}) = 0$, then the correspondence sequence

$$0 \to \Gamma_\Phi(X;\mathcal{A}) \to \Gamma_\Phi(X;\mathcal{B}) \to \Gamma_\Phi(X;\mathcal{C}) \to 0$$

is exact.

Theorem 4.5 Suppose X is a topological space, Φ is a family of support on X, and $\mathcal{A}$ is a sheaf on X. Then

$$H^n_\Phi(X;\mathcal{A}) = 0, \quad n \geq 1$$

if one of the followings verifies:

1. $\mathcal{A}$ is flasque;
2. Φ is paracompactified and $\mathcal{A}$ is Φ-soft.

Proof. Consider the exact sequence

$$0 \to \mathcal{A} \to \mathcal{C}^0(X;\mathcal{A}) \to \mathcal{C}^1(X;\mathcal{A}) \to \cdots$$

Using theorem 3.5 if the first condition verifies and theorem 3.18 if the second, we see that the sequence

$$0 \to \Gamma_\Phi(\mathcal{A}) \to C^0_\Phi(X;\mathcal{A}) \to C^1_\Phi(X;\mathcal{A}) \to \cdots$$

is exact, implying that the higher cohomology groups are all trivial. □

As the Godement resolution is not always easy to compute, we try to determine the cohomology groups using other resolutions.

Suppose X is a topological space, Φ is a family of support on X and $\mathcal{F}^*$ is a differential sheaf on X. Consider the bigraded group

$$K = K(\mathcal{F}^*) = \left\{C^p_\Phi(X;\mathcal{F}^q)\right\}.$$

We can make this into a double complex by taking differentials

$$d' : C^p_\Phi(X;\mathcal{F}^q) \to C^{p+1}_\Phi(X;\mathcal{F}^q)$$

given by the Godement resolution and

$$d'' : C^p_\Phi(X;\mathcal{F}^q) \to C^p_\Phi(X;\mathcal{F}^{q+1})$$

induced from the differential of $\mathcal{F}^*$, up to $(-1)^p$, which satisfies

$$d'd'' + d''d' = 0.$$

Then $d = d' + d''$ defines a differential on the total complex K_{tot}, which is given by

$$K^n_{\text{tot}} = \sum_{p+q=n} C^p_\Phi(X;\mathcal{F}^q).$$

Now consider the spectral sequences given by K:

$${}^{I}E_2^{p,q} = H^p_{d'}(H^q_{d''}(K)), \quad {}^{II}E_2^{p,q} = H^p_{d''}(H^q_{d'}(K)).$$

Since C^p_Φ is exact, we see that

$$(H^q_{d''}(K))^p = H^q(K^{p,*}) = H^q(C^p_\Phi(X;\mathcal{F}^*)) = C^p_\Phi(X;\mathcal{H}^q(\mathcal{F}^*)),$$

and hence

$${}^{I}E_2^{p,q} = H^p(C^*_\Phi(X;\mathcal{H}^q(\mathcal{F}^*))) = H^p_\Phi(X;\mathcal{H}^q(\mathcal{F}^*)).$$

At the same time,

$$(H^q_{d'}(K))^p = H^q(K^{*,p}) = H^q(C^*_\Phi(X;\mathcal{F}^p)) = H^q_\Phi(X;\mathcal{F}^p),$$

implying that

$${}^{II}E_2^{p,q} = H^p(H^q_\Phi(X;\mathcal{F}^*)).$$

Since $K^{p,q} = 0$ for $p < 0$, the second filtration of K, given by

$${}^{II}K^p = \sum_{i\in\mathbb{Z}, j\geq p} K^{i,j},$$

is regular, meaning that for each $n \in \mathbb{Z}$, there exists p_n such that

$$K_{\mathrm{tot}}^n \cap {}^{II}K^p = 0, \quad p \geq p_n.$$

Noting that

$${}^{II}E_2^{p,0} = H^p(H_\Phi^0(X;\mathcal{F}^*)) = H^p(\Gamma_\Phi(X;\mathcal{F}^*))$$

in a canonical way, we have the induced homomorphism

$$H^p(\Gamma_\Phi(\mathcal{F}^*)) \to H^p(K_{\mathrm{tot}}^*).$$

When ${}^{II}E_2^{p,q} = 0$ for $q > 0$, then the above homomorphism is actually isomorphism. Consider the convergence of the spectral sequence given by the first filtration, we obtain the following theorem.

Theorem 4.6 Suppose X is a topological space, Φ is a family of support on X, and $\mathcal{F}^*$ is a differential sheaf on X. Suppose the complexes $H_\Phi^q(X;\mathcal{F}^*)$ are exact for $q > 0$, then we have the convergence of the spectral sequence

$$E_2^{p,q} = H_\Phi^p(X;\mathcal{H}^q(\mathcal{F}^*)) \Rightarrow H^{p+q}(\Gamma_\Phi(\mathcal{F}^*)).$$

Now consider a resolution of a sheaf $\mathcal{A}$ on X:

$$0 \to \mathcal{A} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots.$$

The corresponding double complex $K(\mathcal{F}^*)$ then concentrates in the first quadrant. We have the injective homomorphisms of chain complexes

$$C_\Phi^*(X;\mathcal{A}) \to K_{\mathrm{tot}}^* \leftarrow \Gamma_\Phi(\mathcal{F}^*),$$

which induces homomorphisms of cohomology groups

$$H_\Phi^n(X;\mathcal{A}) \to H^n(K_{\mathrm{tot}}^*) \leftarrow H^n(\Gamma_\Phi(\mathcal{F}^*)).$$

These can be identified with the homomorphisms given by the spectral sequences

$${}^{I}E_2^{n,0} \to H^n(K_{\mathrm{tot}}^*) \leftarrow {}^{II}E_2^{n,0},$$

since we can identified $\mathcal{A}$ with $\mathcal{H}^0(\mathcal{F}^*)$ and $\Gamma_\Phi(\mathcal{F}^*)$ with $H_\Phi^0(X;\mathcal{F}^*)$. Since $\mathcal{F}^*$ is a resolution of $\mathcal{A}$, we have $\mathcal{H}^q(\mathcal{F}^*) = 0$ for $q > 0$, and hence

$${}^{I}E_2^{p,q} = H_\Phi^p(X;\mathcal{H}^q(\mathcal{F}^*)) = 0, \quad q > 0.$$

Hence the homomorphism $H_\Phi^n(X;\mathcal{A}) \to H^n(K_{\mathrm{tot}}^*)$ is actually bijective. We then obtain a homomorphism

$$H^n(\Gamma_\Phi(\mathcal{F}^*)) \to H_\Phi^n(X;\mathcal{A}).$$

Acturally, this result can be viewed as a special case for the following convergence of the spectral sequence given by the second filtration

$${}^{II}E_2^{p,q} = H^p(H_\Phi^q(X;\mathcal{F}^*)) \Rightarrow H_\Phi^{p+q}(X;\mathcal{A}).$$

Together with the convergence of the spectral sequence given by the first filtration (theorem 4.6), we obtain the following isomorphism.

Theorem 4.7 Suppose X is a topological space, Φ is a family of support on X, and $\mathcal{F}^*$ is a resolution of a sheaf $\mathcal{A}$ on X. If $H^q_\Phi(X;\mathcal{F}^*)$ is exact for $q > 0$, then the canonical homomorphism

$$H^n(\Gamma_\Phi(\mathcal{F}^*)) \to H^n_\Phi(X;\mathcal{A})$$

is bijective.

Corollary 4.8 Suppose X is a topological space, Φ is a family of support on X, and $\mathcal{F}^*$ is a resolution of a sheaf $\mathcal{A}$ on X. Then we have the canonical isomorphism

$$H^n_\Phi(X;\mathcal{A}) = H^n(\Gamma_\Phi(\mathcal{F}^*)),$$

if one of the followings verifies:

1. $\mathcal{F}^q$ is flasque for each q;
2. Φ is paracompactified and $\mathcal{F}^q$ is Φ-soft for each q.

Example 4.1 Suppose M is an n-dimensional smooth manifold. They we have the resolution of $\underline{\mathbb{R}}$ on M given by

$$0 \to \underline{\mathbb{R}} \to \Omega^0 \to \Omega^1 \to \cdots.$$

Suppose Φ is a paracompactified family on M. Since C^∞_M is a Φ-soft sheaf of rings, theorem 3.19 implies that Ω^p is Φ-soft for each p. Thus we have the isomorphism

$$H^n_\Phi(M;\underline{\mathbb{R}}) = H^n(\Gamma_\Phi(\Omega^*)) = H^n_{\mathrm{dR},\Phi}(M).$$

The homomorphism from $H^n(\Gamma_\Phi(\mathcal{F}^*))$ to $H^n_\Phi(X;\mathcal{A})$ is actually canonical. This is what the following theorem means.

Theorem 4.9 Suppose X is a topological space, Φ is a family of support on X, and $\mathcal{F}^*$ and $\mathcal{G}^*$ are resolutions of the sheaves $\mathcal{A}$ and $\mathcal{B}$ on X, respectively. If $f : \mathcal{A} \to \mathcal{B}$ is a morphism of sheaf and $g : \mathcal{F}^* \to \mathcal{G}^*$ is compatible with f, then the following diagram commutes:

$$\begin{array}{ccc} H^n(\Gamma_\Phi(\mathcal{F}^*)) & \longrightarrow & H^n_\Phi(X;\mathcal{A}) \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ H^n(\Gamma_\Phi(\mathcal{G}^*)) & \longrightarrow & H^n_\Phi(X;\mathcal{B}) \end{array}$$

The homomorphism also commutes with the connecting homomorphism of a long exact sequence.

Theorem 4.10 Suppose X is a topological space, Φ is a paracompactified family on X, and $\mathcal{F}^*, \mathcal{G}^*, \mathcal{K}^*$ are resolutions of the sheaves $\mathcal{A}, \mathcal{B}, \mathcal{C}$ on X respectively. If we have the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{C} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}^* & \longrightarrow & \mathcal{G}^* & \longrightarrow & \mathcal{K}^* & \longrightarrow & 0 \end{array}$$

and the corresponding sequence

$$0 \to \Gamma_\Phi(\mathcal{F}^q) \to \Gamma_\Phi(\mathcal{G}^q) \to \Gamma_\Phi(\mathcal{K}^q) \to 0$$

is exact for each q, then the following diagram commutes:

$$\begin{array}{ccc} H^n(\Gamma_\Phi(\mathcal{K}^*)) & \longrightarrow & H^n_\Phi(X;\mathcal{C}) \\ \tilde{\partial}\downarrow & & \downarrow\partial \\ H^{n+1}(\Gamma_\Phi(\mathcal{F}^*)) & \longrightarrow & H^{n+1}_\Phi(X;\mathcal{A}) \end{array}$$

4.4 Characterisation of cohomology groups

An interesting thing about the cohomology groups is that they can be determined (up to a natural isomorphism) by some of their properties.

Theorem 4.11 Suppose X is a topological space, Φ is a family of support on X, and

$$F^n : \mathsf{Sh}(X, \mathsf{Ab}) \to \mathsf{Ab}, \quad n = 0, 1, \cdots$$

are functors satisfying the following properties:

1. there is a natural isomorphism
$$\alpha : \Gamma_\Phi \to F^0;$$
2. for each exact sequence
$$0 \to \mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C} \to 0$$
of sheaves on X, we have a natural connecting homomorphism
$$\partial : F^n(\mathcal{C}) \to F^{n+1}(\mathcal{A})$$
yielding the following long exact sequence:
$$0 \to F^0(\mathcal{A}) \xrightarrow{f^*} F^1(\mathcal{B}) \xrightarrow{g^*} F^2(\mathcal{C}) \xrightarrow{\partial} F^1(X;\mathcal{A}) \to \cdots$$
$$\cdots \to F^{n-1}(X;\mathcal{C}) \xrightarrow{\partial} F^n(X;\mathcal{A}) \xrightarrow{f^*} F^n(X;\mathcal{B}) \xrightarrow{g^*} F^n(X;\mathcal{C}) \xrightarrow{\partial} F^{n+1}(X;\mathcal{A}) \to \cdots;$$
3. we have
$$F^n(\mathcal{A}) = 0, \quad n \geq 1$$
whenever $\mathcal{A}$ is flasque.

Then there exist natural isomorphisms

$$T^n : H^n_\Phi(X;-) \to F^n$$

compatible with connecting homomorphisms.

Proof. First we have a natural isomorphism

$$T^0 : H^0_\Phi(X;-) \to F^n$$

given by α and the natural isomorphism from Γ_Φ to $H^0_\Phi(X;-)$. Next we try to construct the natural isomorphisms inductively. Consider the exact sequence of sheaves

$$0 \to \mathcal{A} \to \mathcal{C}^0(X;\mathcal{A}) \to \mathcal{Z}^1(X;\mathcal{A}) \to 0.$$

Then we have the commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \Gamma_\Phi(\mathcal{A}) & \longrightarrow & C^0_\Phi(X;\mathcal{A}) & \longrightarrow & \Gamma_\Phi(\mathcal{Z}^1(X;\mathcal{A})) & \longrightarrow & H^1_\Phi(X;\mathcal{A}) & \longrightarrow H^1_\Phi(X;\mathcal{C}^0(X;\mathcal{A})) \\
& \downarrow & & \downarrow & & \downarrow & & & \\
0 \longrightarrow & F^0(\mathcal{A}) & \longrightarrow & F^0(\mathcal{C}^0(X;\mathcal{A})) & \longrightarrow & F^0(\mathcal{Z}^1(X;\mathcal{A})) & \longrightarrow & F^1(\mathcal{A}) & \longrightarrow F^1(\mathcal{C}^0(X;\mathcal{A}))
\end{array}$$

The vertical arrows are given by α, and since $\mathcal{X}^0(X;\mathcal{A})$ is flasque, it is true that

$$H^1_\Phi(X;\mathcal{C}^0(X;\mathcal{A})) = F^1(\mathcal{C}^0(X;\mathcal{A})) = 0.$$

Therefore there is an isomorphism

$$T^1(\mathcal{A}) : H^1_\Phi(X;\mathcal{A}) \to F^1(\mathcal{A})$$

making the diagram commute. For a morphism $f : \mathcal{A} \to \mathcal{B}$, consider the commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathcal{A} & \longrightarrow & C^0_\Phi(X;\mathcal{A}) & \longrightarrow & \mathcal{Z}^1(X;\mathcal{A}) & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathcal{B} & \longrightarrow & C^0_\Phi(X;\mathcal{B}) & \longrightarrow & \mathcal{Z}^1(X;\mathcal{B}) & \longrightarrow 0
\end{array}$$

which actually implies the naturality of T^1. It is clear that T^1 commutes with the connecting homomorphisms. Suppose we have constructed the natural isomorphism

$$T^n : H^n_\Phi(X;-) \to F^n,$$

with $n \geq 1$, commuting with connecting homomorphisms. We have the commutative diagram of exact sequences:

$$\begin{array}{ccccccc}
H^n_\Phi(X;\mathcal{C}^0(X;\mathcal{A})) & \longrightarrow & H^n_\Phi(\mathcal{Z}^1(X;\mathcal{A})) & \longrightarrow & H^{n+1}_\Phi(X;\mathcal{A}) & \longrightarrow & H^{n+1}_\Phi(X;\mathcal{C}^0(X;\mathcal{A})) \\
& & {\scriptstyle T^n}\downarrow & & & & \\
F^n(\mathcal{C}^0(X;\mathcal{A})) & \longrightarrow & F^n(\mathcal{Z}^1(X;\mathcal{A})) & \longrightarrow & F^{n+1}(\mathcal{A}) & \longrightarrow & F^{n+1}(\mathcal{C}^0(X;\mathcal{A}))
\end{array}$$

The flasque property of $\mathcal{C}^0(X;\mathcal{A})$ implies that

$$H^n_\Phi(X;\mathcal{C}^0(X;\mathcal{A})) = F^n(\mathcal{C}^0(X;\mathcal{A})) = H^{n+1}_\Phi(X;\mathcal{C}^0(X;\mathcal{A})) = F^{n+1}(\mathcal{C}^0(X;\mathcal{A})) = 0.$$

We then obtain an isomorphism

$$T^{n+1}(\mathcal{A}) : H^{n+1}_\Phi(X;\mathcal{A}) \to F^{n+1}(\mathcal{A}).$$

The naturality and the commutativity with the connecting homomorphisms of T^{n+1} are both direct from our construction. □

5 Čech cohomology

Through this section, we consider presheaves and sheaves of abelian groups on a fixed topological space X. If $\mathcal{A}$ is a presheaf, we always suppose that $\mathcal{A}(\emptyset) = 0$.

5.1 Cohomology with respect to an open cover

Suppose $\mathcal{A}$ is a presheaf on X and $\mathfrak{U} = \{U_i\}_{i \in I}$ is an open cover of X. For a subset $s = \{i_0, \cdots, i_p\}$ of I, define

$$U_s = U_{i_0 \cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}.$$

Let

$$C^p(\mathfrak{U}; \mathcal{A}) = \prod_{\substack{s=\{i_0,\cdots,i_p\} \subset I \\ U_s \neq \emptyset}} \mathcal{A}(U_s)$$

be the group of **$\mathcal{A}$-valued cochains of degree p of $\mathfrak{U}$**. A cochain $\alpha \in C^p(\mathfrak{U}; \mathcal{A})$ has the form

$$\alpha = (\alpha_{i_0 \cdots i_p})_{i_0, \cdots, i_p \in I},$$

with

$$\alpha_{i_0 \cdots i_p} \in \mathcal{A}(U_{i_0 \cdots i_p}).$$

Define the differential

$$d : C^p(\mathfrak{U}; \mathcal{A}) \to C^{p+1}(\mathfrak{U}; \mathcal{A})$$

by

$$(d\alpha)_{i_0 \cdots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0 \cdots \hat{i}_k \cdots i_{p+1}} |_{U_{i_0 \cdots i_{p+1}}}.$$

This is similar to the differential of a singular cochain. We denote the cohomology group of $C^*(\mathfrak{U}; \mathcal{A})$ of degree n by $H^n(\mathfrak{U}; \mathcal{A})$.

For a morphism $f : \mathcal{A} \to \mathcal{B}$ of presheaves, we have a induced cochain homomorphism $C^*(\mathfrak{U}; \mathcal{A}) \to C^*(\mathfrak{U}; \mathcal{B})$ and then a homomorphism

$$f^* : H^n(\mathfrak{U}; \mathcal{A}) \to H^n(\mathfrak{U}; \mathcal{B})$$

for each $n \geq 0$.

Consider an open cover $\mathfrak{U} = \{U_i\}_{i \in I}$ of X and an open subset $V \subset X$.Then the family of open sets given by $\mathfrak{U} \cap V = \{U_i \cap V\}_{i \in I}$ is an open cover of V. We take the notation that

$$C^*(\mathfrak{U} \cap V; \mathcal{A}) = C^*(\mathfrak{U} \cap V; \mathcal{A}|_V).$$

For each open sets $W \subset V \subset X$, the restriction of sections induces a morphism

$$C^*(\mathfrak{U} \cap V; \mathcal{A}) \to C^*(\mathfrak{U} \cap W; \mathcal{A}).$$

It follows that the assignment

$$V \mapsto C^n(\mathfrak{U} \cap V; \mathcal{A})$$

gives a presheaf on $\mathcal{A}$, denoted by $\mathcal{C}^n(\mathfrak{U}; \mathcal{A})$. We then also obtain a differential presheaf

$$\mathcal{C}^*(\mathfrak{U}; \mathcal{A}) = \{\mathcal{C}^n(\mathfrak{U}; \mathcal{A})\}_{n \geq 0}.$$

If $\mathcal{A}$ is a sheaf, then each $\mathcal{C}^n(\mathfrak{U}; \mathcal{A})$ is a sheaf and $\mathcal{C}^*(\mathfrak{U}; \mathcal{A})$ is a differential sheaf. We can see that

$$C^*(\mathfrak{U}; \mathcal{A}) = \Gamma(\mathcal{C}^*(\mathfrak{U}; \mathcal{A})),$$

which inspires us to define

$$C^*_\Phi(\mathfrak{U}; \mathcal{A}) = \Gamma_\Phi(\mathcal{C}^*(\mathfrak{U}; \mathcal{A})),$$

where Φ is a family of support on X. The cohomology groups of $C^*_\Phi(\mathfrak{U};\mathcal{A})$ are denoted by $H^n_\Phi(\mathfrak{U};\mathcal{A})$, with $n \geq 0$.

Note that we have a sheaf morphism

$$j : \mathcal{A} \to \mathcal{C}^0(\mathfrak{U};\mathcal{A})$$

given by

$$j(\alpha)_i = \alpha|_{U_i \cap V}, \quad \alpha \in \mathcal{A}(V), i \in I.$$

The axiom of sheaf implies that j is injective with $\mathrm{im}(j) = \ker(d)$.

Theorem 5.1 Suppose X is a topological space and $\mathfrak{U}$ is an open cover of X. Then for each sheaf $\mathcal{A}$ on X the following sequence is exact

$$0 \to \mathcal{A} \xrightarrow{j} \mathcal{C}^0(\mathfrak{U};\mathcal{A}) \xrightarrow{d} \mathcal{C}^1(\mathfrak{U};\mathcal{A}) \xrightarrow{d} \mathcal{C}^2(\mathfrak{U};\mathcal{A}) \to \cdots,$$

i.e., $\mathcal{C}^*(\mathfrak{U};\mathcal{A})$ is a resolution of $\mathcal{A}$.

Proof. We have shown the exactness at $\mathcal{A}$ and $\mathcal{C}^0(\mathfrak{U};\mathcal{A})$, so it remains to show $\mathrm{im}(d) = \ker(d)$ for $n > 0$. Consider $x \in X$ and a germ in $\ker(d)$ defined by $\alpha \in C^n(\mathfrak{U} \cap V;\mathcal{A})$, where V is a neighborhood of x. Since $\mathfrak{U}$ is an open cover, we may assume $V \subset U_k$ for some index $j \in I$. Then

$$V \cap U_{ji_0\cdots i_{n-1}} = V \cap U_{i_0\cdots i_{n-1}}$$

for any $i_0, \cdots, i_{n-1} \in I$. Define $\beta \in C^{n-1}(\mathfrak{U} \cap V;\mathcal{A})$ by

$$\beta_{i_0\cdots i_{n-1}} = \alpha_{ji_0\cdots i_{n-1}}, \quad i_0, \cdots, i_{n-1} \in I.$$

Since $d\alpha = 0$, we have

$$0 = (d\alpha)_{ji_0\cdots i_n} = \alpha_{i_0\cdots i_n} - \sum_{k=0}^{n} (-1)^k \alpha_{ji_0\cdots \hat{i}_k \cdots i_n},$$

implying that

$$(d\beta)_{i_0\cdots i_n} = \sum_{k=0}^{n} (-1)^k \beta_{i_0\cdots \hat{i}_k \cdots i_n} = \sum_{k=0}^{n} (-1)^k \alpha_{ji_0\cdots \hat{i}_k \cdots i_n} = \alpha_{i_0\cdots i_n}.$$

It follows that $\alpha = d\beta$ and the exactness at $\mathcal{C}^n(\mathfrak{U};\mathcal{A})$ follows. □

Corollary 5.2 Suppose $\mathfrak{U}$ is an open cover of a topological space X and Φ is a family of support on X. Then for each sheaf $\mathcal{A}$ on X, we have the canonical isomorphism

$$H^0_\Phi(\mathfrak{U};\mathcal{A}) = \Gamma_\Phi(\mathcal{A}) = H^0_\Phi(X;\mathcal{A}).$$

Proof. Consider the exact sequence

$$0 \to \mathcal{A} \to \mathcal{C}^0(\mathfrak{U};\mathcal{A}) \to \mathcal{C}^1(\mathfrak{U};\mathcal{A}).$$

Since Γ_Φ is left exact, we have the exact sequence

$$0 \to \Gamma_\Phi(\mathcal{A}) \to C^0_\Phi(\mathfrak{U};\mathcal{A}) \to C^1_\Phi(\mathfrak{U};\mathcal{A}).$$

It follows that $H^0_\Phi(\mathfrak{U};\mathcal{A})$ is isomorphic to $\Gamma_\Phi(\mathcal{A})$ canonically. □

Theorem 5.3 Suppose $\mathfrak{U}$ is an open cover of a topological space X, Φ is a family of support on X, and $\mathcal{A}$ is a sheaf on X. If $\mathcal{A}$ is flasque, then

$$H^n_\Phi(\mathfrak{U};\mathcal{A}) = 0, \quad n \geq 1.$$

Proof. By theorem 5.1 and 3.5, it remains to verify the flasque property for each $C^n(\mathfrak{U};\mathcal{A})$, which is direct from the definition. □

Applying the results in section 4.3, we obtain a canonical homomorphism

$$H^*_\Phi(\mathfrak{U};\mathcal{A}) \to H^*_\Phi(X;\mathcal{A})$$

for each open cover $\mathfrak{U}$ of X, each family of support Φ and each sheaf $\mathcal{A}$ on X.

5.2 Relations between the cohomology with respect to an open cover and that of the whole space

Suppose $\mathfrak{U}$ is an open cover of a topological space X and $\mathcal{F}^* = \{\mathcal{F}^n\}$ is a differential sheaf on X. As shown in section 4.3, we can consider the double complex

$$K = K(\mathfrak{U};\mathcal{F}^*) = \left\{C^p(\mathfrak{U};\mathcal{F}^q)\right\},$$

with the differentials

$$d' : C^p(\mathfrak{U};\mathcal{F}^q) \to C^{p+1}(\mathfrak{U};\mathcal{F}^q), \quad d'' : C^p(\mathfrak{U};\mathcal{F}^q) \to C^p(\mathfrak{U};\mathcal{F}^{q+1}).$$

To determine the corresponding spectral sequences, note that

$$(H^q_{d'}(K))^p = H^q(K^{*,p}) = H^q(C^*(\mathfrak{U};\mathcal{F}^p)) = H^q(\mathfrak{U};\mathcal{F}^p),$$

and then

$${}^{II}E_2^{p,q} = H^p(H^q(\mathfrak{U};\mathcal{F}^*)).$$

For ${}^{I}E_2$, we have

$$\begin{aligned}(H^q_{d''}(K))^p = H^q(K^{p,*}) &= H^q(C^p(\mathfrak{U};\mathcal{F}^*)) \\ &= H^q\left(\prod_{s=\{i_0,\cdots,i_p\}\subset I} \mathcal{F}^*(U_s)\right) \\ &= \prod_{s=\{i_0,\cdots,i_p\}\subset I} H^q(\mathcal{F}^*(U_s)) \\ &= \prod_{s=\{i_0,\cdots,i_p\}\subset I} \mathcal{H}^q(\mathcal{F}^*)(U_s) \\ &= C^p(\mathfrak{U};\mathcal{H}^q(\mathcal{F}^*)),\end{aligned}$$

implying that

$${}^{I}E_2^{p,q} = H^p(C^*(\mathfrak{U};\mathcal{H}^q(\mathcal{F}^*))) = H^p(\mathfrak{U};\mathcal{H}^q(\mathcal{F}^*)).$$

Now suppose $\mathcal{F}^* = \mathcal{C}^*(X;\mathcal{A})$ is the Godement resolution of a sheaf $\mathcal{A}$. The canonical injection $\mathcal{A} \to \mathcal{F}^0$ then induces a canonical homomorphism of complexes

$$C^*(\mathfrak{U};\mathcal{A}) \to C^*(\mathfrak{U};\mathcal{F}^0) \to K^*_{\text{tot}},$$

which further induces a homomorphism

$$H^n(\mathfrak{U};\mathcal{A}) \to H^n(K^*_{\mathrm{tot}})$$

of cohomology groups. As $\mathcal{F}^*$ is a flasque resolution of $\mathcal{A}$, we have $H^q(\mathfrak{U};\mathcal{A}) = 0$ for $q > 0$ by theorem 5.3, and hence

$$^{II}E_2^{p,q} = H^p(\mathfrak{U};\mathcal{H}^q(\mathcal{F}^*)) = 0, \quad q > 0.$$

It follows that the homomorphisms

$$H^n(X;\mathcal{A}) = H^n(\Gamma(\mathcal{F}^*)) = H^n(H^0(\mathfrak{U};\mathcal{F}^*)) \to H^n(K^*_{\mathrm{tot}})$$

are all bijective. We then obtain natural homomorphisms

$$H^n(\mathfrak{U};\mathcal{A}) \to H^n(X;\mathcal{A}).$$

Turning to the other spectral sequence, we find the following theorem.

Theorem 5.4 Suppose $\mathfrak{U}$ is an open cover of a topological space X and $\mathcal{A}$ is a sheaf on X. Then the following convergence of the spectral sequence holds:

$$E_2^{p,q} = H^p(\mathfrak{U};\mathcal{H}^q(\mathcal{A})) \Rightarrow H^{p+q}(X;\mathcal{A}),$$

where the sheaf $\mathcal{H}^q(\mathcal{A})$ is given by

$$U \mapsto H^q(\mathcal{C}^*(X;\mathcal{A})(U))$$

for each $q \geq 0$, with $\mathcal{C}^*(X;\mathcal{A})$ the Godement resolution of $\mathcal{A}$.

Corollary 5.5 Suppose $\mathfrak{U} = \{U_i\}_{i\in I}$ is an open cover of a topological space X and $\mathcal{A}$ is a sheaf on X. If

$$H^q(\mathcal{C}^*(X;\mathcal{A})(U_s)) = 0, \quad s \subset I, q > 0,$$

then the canonical homomorphisms

$$H^n(\mathfrak{U};\mathcal{A}) \to H^n(X;\mathcal{A})$$

are bijective.

The above results can actually be generalized to H^n_Φ, which yields canonical homomorphisms

$$H^n_\Phi(\mathfrak{U};\mathcal{A}) \to H^n_\Phi(X;\mathcal{A}).$$

It can be seen from our constructions that these homomorphisms are the same as those at the end of section 5.1.

5.3 Čech cohomology

Suppose $\mathfrak{U} = \{U_i\}_{i\in I}$ and $\mathfrak{V} = \{V_j\}_{j\in J}$ are open covers of a topoological space X such that $\mathfrak{V}$ is a refinement of $\mathfrak{U}$, i.e., for each $j \in J$, there is $i \in I$, such that $V_j \subset U_i$. Then we have a map $\iota : J \to I$ such that $V_j \subset U_{\iota(j)}$ for each j. Consider a family of support Φ and a sheaf $\mathcal{A}$ on X. We can define a homomorphism of complexes

$$\iota^* : C^*_\Phi(\mathfrak{U};\mathcal{A}) \to C^*_\Phi(\mathfrak{V};\mathcal{A})$$

by

$$(\iota^*(\alpha))_{j_0\cdots j_n} = \alpha_{\iota(j_0)\cdots\iota(j_n)}|_{V_{j_0\cdots j_n}}, \quad j_0, \cdots, j_n \in J, \alpha \in C^n_\Phi(\mathfrak{U};\mathcal{A}).$$

This further induces a homomorphisms between cohomology groups. However, since the choice of ι is not unique, the naturality of the homomorphisms need considering.

Theorem 5.6 Suppose $\mathfrak{U} = \{U_i\}_{i\in I}$ and $\mathfrak{V} = \{V_j\}_{j\in J}$ are open covers of a topoological space X such that $\mathfrak{V}$ is a refinement of $\mathfrak{U}$, Φ is a family of support on X, and $\mathcal{A}$ is a sheaf on X. If ι_1 and ι_2 are two maps from J to I such that $V_j \subset U_{\iota_1(j)}$ and $V_j \subset U_{\iota_2(j)}$ for each $j \in J$, then the induced homomorphisms

$$\iota_1^*, \iota_2^* : C_\Phi^*(\mathfrak{U}; \mathcal{A}) \to C_\Phi^*(\mathfrak{V}; \mathcal{A})$$

are homotopic.

Proof. Define homomorphisms

$$K : C_\Phi^n(\mathfrak{U}; \mathcal{A}) \to C_\Phi^{n-1}(\mathfrak{V}; \mathcal{A})$$

for each $n > 0$ by

$$(K\alpha)_{j_0\cdots j_{n-1}} = \sum_{k=0}^{n-1} (-1)^k \alpha_{\iota_1(j_0)\cdots\iota_1(j_k)\iota_2(j_k)\cdots\iota_2(j_{n-1})}|_{V_{j_0\cdots j_{n-1}}}.$$

We can verify that

$$\iota_2^* - \iota_1^* = dK + Kd,$$

i.e., ι_1^* and ι_2^* are homotopic through K. □

Since homotopic cochain complex homomorphisms induce the same homomorphism of cohomology groups, we obtain a canonical homomorphism

$$H_\Phi^n(\mathfrak{U}; \mathcal{A}) \to H_\Phi^n(\mathfrak{V}; \mathcal{A})$$

for each $n \geq 0$. Moreover, if $\mathfrak{W}$ is a refinement of $\mathfrak{V}$, then the following diagram commutes:

$$\begin{array}{ccc} H_\Phi^n(\mathfrak{U}; \mathcal{A}) & \longrightarrow & H_\Phi^n(\mathfrak{W}; \mathcal{A}) \\ & \searrow \quad \nearrow & \\ & H_\Phi^n(\mathfrak{V}; \mathcal{A}) & \end{array}$$

We also have the commutative diagram

$$\begin{array}{ccc} H_\Phi^n(\mathfrak{U}; \mathcal{A}) & \longrightarrow & H_\Phi^n(\mathfrak{V}; \mathcal{A}) \\ & \searrow \quad \swarrow & \\ & H_\Phi^n(X; \mathcal{A}) & \end{array}$$

Consider the collection $\mathfrak{C}(X)$ of open covers of X of the form $\tilde{\mathfrak{U}} = \{\tilde{U}_x\}_{x\in X}$ such that $x \in \tilde{U}_x$ for each x. Equip $\mathfrak{C}(X)$ with the partial order given by

$$\tilde{\mathfrak{U}} \leq \tilde{\mathfrak{V}} \quad \Longleftrightarrow \quad \tilde{U}_x \subset \tilde{V}_x \text{ for all } x \in X.$$

For $\tilde{\mathfrak{U}} \leq \tilde{\mathfrak{V}}$, the identity map on X induces a canonical homomorphism

$$C_\Phi^*(\tilde{\mathfrak{V}}; \mathcal{A}) \to C_\Phi^*(\tilde{\mathfrak{U}}; \mathcal{A})$$

for each family of support Φ on X. Define the **Čech cochain complex** on X by

$$\check{C}_\Phi^*(X; \mathcal{A}) = \varinjlim C_\Phi^*(\tilde{\mathfrak{U}}; \mathcal{A}),$$

where the inductive limit is taken over $\mathfrak{U} \in \mathfrak{C}(X)$. The **Čech homology groups** of $\mathcal{A}$ is then defined to be

$$\check{H}_\Phi^n(X; \mathcal{A}) = H^n(\check{C}_\Phi^*(X; \mathcal{A})), \quad n \geq 0.$$

We can see that

$$\tilde{H}^n_\Phi(X;\mathcal{A}) = H^n(\varinjlim C^*_\Phi(\tilde{\mathfrak{U}};\mathcal{A})) = \varinjlim H^n(C^*_\Phi(\tilde{\mathfrak{U}};\mathcal{A})) = \varinjlim H^n_\Phi(\tilde{\mathfrak{U}};\mathcal{A}).$$

Suppose $\mathfrak{U}$ is an arbitrary open cover of X. There exists $\tilde{\mathfrak{U}} \in \mathfrak{C}(X)$ such that $\tilde{\mathfrak{U}}$ is a refinement of $\mathfrak{U}$. Then we have a canonical homomorphism given by the refinement relation

$$H^n_\Phi(\mathfrak{U};\mathcal{A}) \to H^n_\Phi(\tilde{\mathfrak{U}};\mathcal{A}),$$

and a canonical homomorphism given by the inductive limit

$$H^n_\Phi(\tilde{\mathfrak{U}};\mathcal{A}) \to \tilde{H}^n_\Phi(X;\mathcal{A}).$$

Their composition gives a homomorphism

$$H^n_\Phi(\mathfrak{U};\mathcal{A}) \to \tilde{H}^n_\Phi(X;\mathcal{A}).$$

We claim that this homomorphism is independent of the choice of $\tilde{\mathfrak{U}}$. Indeed, suppose $\tilde{\mathfrak{V}} \in \mathfrak{C}(X)$ is another refinement of $\mathfrak{U}$. Then we have another open cover $\tilde{\mathfrak{W}} \in \mathfrak{C}(X)$ of X such that $\tilde{\mathfrak{W}} \le \tilde{\mathfrak{U}}$ and $\tilde{\mathfrak{W}} \le \tilde{\mathfrak{V}}$. It follows that we have the commutative diagram

$$\begin{array}{ccccc} & & H^n_\Phi(\mathfrak{U};\mathcal{A}) & & \\ & \swarrow & \downarrow & \searrow & \\ H^n_\Phi(\tilde{\mathfrak{U}};\mathcal{A}) & \longrightarrow & H^n_\Phi(\tilde{\mathfrak{W}};\mathcal{A}) & \longleftarrow & H^n_\Phi(\tilde{\mathfrak{V}};\mathcal{A}) \\ & \searrow & \downarrow & \swarrow & \\ & & \tilde{H}^n_\Phi(X;\mathcal{A}) & & \end{array}$$

This shows that the above homomorphism is canonical.

It can be shown that for each open covers $\mathfrak{U}$ and $\mathfrak{V}$ of X such that $\mathfrak{V}$ is a refinement of $\mathfrak{U}$, we have the commutative diagram

$$\begin{array}{ccc} H^n_\Phi(\mathfrak{U};\mathcal{A}) & \longrightarrow & H^n_\Phi(\mathfrak{V};\mathcal{A}) \\ & \searrow \quad \swarrow & \\ & \tilde{H}^n_\Phi(X;\mathcal{A}) & \end{array}$$

Furthermore, by the explicit construction of the inductive limit of abelian groups, we can see that

$$\tilde{H}^n_\Phi(X;\mathcal{A}) = \varinjlim H^n_\Phi(\mathfrak{U};\mathcal{A}),$$

where the inductive limit is taken over all the open covers $\mathfrak{U}$ over X. More generally, if $\mathfrak{C}$ is a family of open covers on X such that each open cover $\mathfrak{U}$ of X attains a refinement in $\mathfrak{C}$, then $\tilde{H}^n_\Phi(X;\mathcal{A})$ is the inductive limit taken over $\mathfrak{C}$.

Theorem 5.7 Suppose X is a topological space, Φ is a family of support on X such that each $S \in \Phi$ attains a neighborhood in Φ. Then the functor $\mathcal{A} \mapsto \tilde{C}^*_\Phi(X;\mathcal{A})$ takes an exact sequence of *presheaves* to an exact sequence of complexes.

Proof. Consider an exact sequence of presheaves

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0,$$

which implies an exact sequence

$$0 \to \mathcal{A}(U) \to \mathcal{B}(U) \to \mathcal{C}(U) \to 0$$

for each open $U \subset X$. It follows that

$$0 \to C^*(\mathfrak{U}; \mathcal{A}) \to C^*(\mathfrak{U}; \mathcal{B}) \to C^*(\mathfrak{U}; \mathcal{C}) \to 0$$

is exact for each open cover $\mathfrak{U}$, which yields from the left exactness of Γ_Φ that

$$0 \to C^*_\Phi(\mathfrak{U}; \mathcal{A}) \to C^*_\Phi(\mathfrak{U}; \mathcal{B}) \to C^*_\Phi(\mathfrak{U}; \mathcal{C})$$

is exact. Passing to the inductive limit, we see the exactness of

$$0 \to \tilde{C}^*_\Phi(X; \mathcal{A}) \to \tilde{C}^*_\Phi(X; \mathcal{B}) \to \tilde{C}^*_\Phi(X; \mathcal{C}).$$

It remains to show the surjectivity of $\tilde{C}^*_\Phi(X; \mathcal{B}) \to \tilde{C}^*_\Phi(X; \mathcal{C})$.

Consider an element in $\tilde{C}^p_\Phi(X; \mathcal{C})$ represented by an cochain $\alpha \in C^p_\Phi(\tilde{\mathfrak{U}}; \mathcal{C})$ with $S \in \Phi$ being its support. Take a neighborhood $T \in \Phi$ of S. By replacing $\tilde{\mathfrak{U}}$ with a refinement if necessary, we may assume that $U_x \subset T$ for each $x \in S$ and $\alpha_s = 0$ if $s = \{x_0, \cdots, x_p\}$ is not contained in S. Then each α_s can be lifted to $\beta_s \in \mathcal{B}(U_s)$, which generates a cochain $\beta \in C^p_\Phi(\tilde{\mathfrak{U}}; \mathcal{B})$. This completes the proof. □

This implies that an exact sequence of presheaves yields an exact sequence of Čech cohomology groups.

Recalling that we have a canonical homomorphism

$$H^n(\mathfrak{U}; \mathcal{A}) \to H^n(X; \mathcal{A})$$

for each open cover $\mathfrak{U}$ of X, the universal property of the inductive limit yields a canonical homomorphism

$$\tilde{H}^n(X; \mathcal{A}) \to H^n(X; \mathcal{A}).$$

The rest of this section devotes to showing the bijectivity of this homomorphism.

Suppose $\mathcal{F}^* = \mathcal{C}^*(X; \mathcal{A})$ is the Godement resolution of $\mathcal{A}$. Consider the double complex

$$\tilde{K} = \left\{\tilde{C}^p(X; \mathcal{F}^q)\right\}.$$

We have the canonical homomorphisms of complexes

$$\tilde{C}^*(X; \mathcal{A}) \to \tilde{K}^*_{\mathrm{tot}} \leftarrow \Gamma(\mathcal{F}^*).$$

By theorem 5.3, we have $\tilde{H}^q(X; \mathcal{F}^p)$ for each $q > 0$ and p, implying that

$$^{II}E_2^{p,q} = H^p(\tilde{H}^q(X; \mathcal{F}^*)) = 0, \quad q > 0.$$

Thus the canonical homomorphism

$$H^n(X; \mathcal{A}) = H^n(\tilde{H}^0(X; \mathcal{F}^*)) \to H^n(\tilde{K}^*_{\mathrm{tot}})$$

is bijective. Define the presheaf

$$\mathcal{H}^q(X; \mathcal{A}) : U \mapsto H^q(\mathcal{F}^*(U)) = H^q(U; \mathcal{A})$$

for each q. Then as $\tilde{C}^p(X; -)$ is an exact functor from $\mathsf{pSh}(X, \mathsf{Ab})$, we have

$$H^q(K^{p,*}) = H^q(\tilde{C}^p(X; \mathcal{F}^*)) = \tilde{C}^p(X; \mathcal{H}^q(X; \mathcal{A})),$$

and hence

$$^{I}E_2^{p,q} = \tilde{H}^p(X; \mathcal{H}^q(X; \mathcal{A})).$$

Theorem 5.8 Suppose X is a topological space and $\mathcal{A}$ is a sheaf on X. Consider the presheaves

$$\mathcal{H}^q(X;\mathcal{A}) : U \mapsto H^q(\mathcal{F}^*(U)).$$

Then the following convergence of the spectral sequence holds:

$$E_2^{p,q} = \tilde{H}^p(X;\mathcal{H}^q(X;\mathcal{A})) \Rightarrow H^{p+q}(X;\mathcal{A}).$$

The spectral sequence actually gives the canonical homomorphism

$$\tilde{H}^n(X;\mathcal{A}) \to H^n(X;\mathcal{A})$$

by the canonical isomorphism $\mathcal{H}^0(X;\mathcal{A}) = \mathcal{A}$. It is worth noting that the sheaf generated by $\mathcal{H}^q(X;\mathcal{A})$ is zero for $q > 0$, as each cocycle of $\mathcal{F}^*$ is a coboundary locally.

Lemma 5.9 If $\mathcal{F}$ is a presheaf on X generating a zero sheaf, then $\tilde{H}^0(X;\mathcal{F}) = 0$.

Proof. Consider a cochain $\alpha \in \tilde{C}^0(X;\mathcal{F})$ given by an open cover $\tilde{\mathfrak{U}} = \{U_x\}_{x\in X}$ of X and a family $\{\alpha_x\}$ with $\alpha_x \in \mathcal{F}(U_x)$. Since the sheaf generated by $\mathcal{F}$ is zero, there is a neighborhood $V_x \subset U_x$ of x such that $\alpha_x|_{V_x} = 0$. Passing to $\tilde{\mathfrak{V}} = \{V_x\}_{x\in X}$, we obtain $\tilde{C}^0(X;\mathcal{F}) = 0$, which clearly implies that $\tilde{H}^0(X;\mathcal{F}) = 0$. □

We then see that $\tilde{H}^0(X;\mathcal{H}^1(X;\mathcal{A})) = \tilde{H}^0(X;\mathcal{H}^2(X;\mathcal{A})) = 0$, which implies the following corollary.

Corollary 5.10 Suppose X is a topological space and $\mathcal{A}$ is a sheaf on X. Then the canonical homomorphism

$$\tilde{H}^n(X;\mathcal{A}) \to H^n(X;\mathcal{A})$$

is bijective for $n = 0, 1$ and injective for $n = 2$.

The above results actually hold for each family of support Φ satisfying the condition of theorem 5.7.

Theorem 5.11 Suppose X is a topological space, Φ is a paracompactified family on X, and $\mathcal{F}$ is a presheaf on X. If the sheaf generated by $\mathcal{F}$ is zero, then

$$\tilde{H}^n_\Phi(X;\mathcal{F}) = 0, \quad n \geq 0.$$

Proof. We will show that each cohomological class in $\tilde{H}^n_\Phi(X;\mathcal{A})$ can be represented by a locally finite open cover $\mathfrak{U}$ and a cocycle on $\mathfrak{U}$. Next we will show that each cochain on this cover induces zero on a refinement.

Consider a cohomological class in $\tilde{H}^n_\Phi(X;\mathcal{A})$ represented by an open cover $\mathfrak{U} = \{U_i\}_{i\in I}$ and a cocycle $\alpha \in C^n(\mathfrak{U};\mathcal{A})$ supported on $S \in \Phi$. Take a neighborhood $S' \in \Phi$ of S, for which we can assume that each U_i intersecting S is contained in S'. Let

$$I_0 = \{i \in I \mid U_i \cap S \neq \emptyset\}.$$

Since α supports on S, each $x \in X \setminus S$ attains a neighborhood $V(x)$ such that α_s induces zero on $U_s \cap V(x)$ for each $s \subset I$. Replacing $\mathfrak{U}$ by its refinement if necessary, we may assume that each U_i with $i \notin I_0$ is contained in some $V(x)$, which implies that $\alpha_s = 0$ if s is not contained in I_0. We then see that the cohomological class can be represented by an open cover $\mathfrak{U} = \{U_i\}_{i\in\{0\}\cup I_0}$ with $U_0 = X \setminus S$ and $U_i \subset S'$ if $i \in I_0$. Since S' is paracompact, there is a locally finite refinement of $\mathfrak{U} \cap S'$. Passing to this refinement, it is valid to assume the local finiteness of $\mathfrak{U}$.

Now consider a locally finite recover $\mathfrak{U} = \{U_i\}_{i\in I}$ together with a refinement $\mathfrak{V} = \{V_i\}_{i\in I}$ with $\overline{V}_i \subset U_i$ for each $i \in I$ and $\alpha \in C^n_\Phi(\mathfrak{U}; \mathcal{A})$. For each $x \in X$ we can take a neighborhood W_x such that $x \in U_i$ implies $W_x \subset U_i$, $x \in V_i$ implies $W_x \subset V_i$, and that W_x intersects V_i implies $x \in U_i$. Moreover, since the sheaf generated by $\mathcal{A}$ is zero, each $x \in U_s$ has a neighborhood on which α_s induces zero. Thus we can meanwhile asssume that $x \in U_s$ implies $\alpha_s|_{W_x} = 0$. Take any map $\iota : X \to I$ such that $W_x \subset V_{\iota(x)}$ for each $x \in X$. Consider any $(x_1, \cdots, x_n) \subset X$ such that $W_{x_0\cdots x_n} \neq \emptyset$. Assume $i_k = \iota(x_k)$ for each k. As

$$W_{x_0} \cap \cdots \cap W_{x_n} \neq \emptyset,$$

W_{x_0} intersects each $W_{x_k} \subset V_{i_k}$, and then our assumption suggests that $x_0 \in U_{i_0\cdots i_n}$. It follows that $W_{x_0} \subset U_{i_0\cdots i_n}$, and $\alpha_{i_0\cdots i_n}|_{W_{x_0}} = 0$. Clearly we obtain

$$\alpha_{i_0\cdots i_n}|_{W_{x_0\cdots x_n}} = 0,$$

i.e., $\iota^*(\alpha) = 0$. □

Theorem 5.12 Suppose X is a topological space, Φ is a paracompactified family on X and $\mathcal{A}$ is a sheaf on X. Then the canonical homomorphisms

$$\tilde{H}^n_\Phi(X; \mathcal{A}) \to H^n_\Phi(X; \mathcal{A})$$

are bijective.

Proof. We have

$$\tilde{H}^p_\Phi(X; \mathcal{H}^q(X; \mathcal{A})) = 0, \quad q > 0$$

from theorem 5.11. Then the generalized version of theorem 5.8 implies the desired isomorphisms. □